Day Ahead Dynamic Pricing for Demand Response in Dynamic Environments

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Abstract— **The problem of optimizing retail pricing of electricity for price-responsive dynamic loads is considered. For the class of day-ahead dynamic prices (DADPs), the problem of retail pricing is modeled as a Stackelberg game with the retailer as the leader and its customers the followers. It is shown that the optimal customer response to a DADP has an affine structure with a deterministic negative definite sensitivity matrix and a stochastic bias. With this structure, tradeoffs between consumer surplus and retail profit can be characterized by a convex region with a concave and non-increasing Pareto front, each point on the Pareto front corresponding to an equilibrium in a dynamic game with a particular payoff function; any consumer surplus-retail profit pair above the Pareto front is not attainable by any dynamic pricing scheme. The optimal DADP that maximizes the social welfare is shown to be that maximizes the consumer surplus thus making retail profit zero. Effects of renewable energy are also considered.**

I. INTRODUCTION

Demand response is one of the key features of a future smart grid. Through pricing and other mechanisms, a carefully designed demand response program offers economic benefits to the consumers and empowers a consumer to manage energy usage actively. Demand response can also provide the flexibility that allows an operator or a retailer to improve operation efficiency, enable greater renewable integration, and increase overall social welfare.

One form of demand response is to enroll consumers to a program that provides reduced price. In return, the consumer allows the retailer to interrupt his services with a certain probability. In such a demand response program, a consumer enjoys price certainty with guaranteed economic benefits, but he faces service uncertainties. Examples of such kind of "emergency demand response" include [1], [2].

A second form of demand response is to let the consumer manage his own energy usage but offer incentives to the consumer through pricing. The retailer determines the price based on the risks it faces, and the consumer responds to prices voluntarily. Here, the consumer has the full control of the level of service, but he faces price uncertainties. Examples of such programs can be found in [3], [4].

We consider the second approach to demand response that allows the consumer to optimize his own demand in response to dynamic pricing that varies from day to day and from hour to hour within each day. In particular, we focus on the class of pricing mechanisms, referred to as day-ahead dynamic pricing (DADP), where the retailer posts day-ahead hourly prices, and these prices will be fixed at the day of consumption. First considered in [3], DADP has been in place for large retail customers for years. The advantage of DADP for a consumer is that the consumer has the price certainty one day ahead of time so that he can plan accordingly based on the posted prices and his desired quality of service.

Although the day ahead pricing reduces price uncertainties to the consumer, it presents nontrivial challenges to the retailer who has to cope with wholesale price fluctuations in real-time and other environmental uncertainties such as weather conditions. To this end, the retailer has to hedge against real-time operating risks by optimizing the retail price accordingly.

A main focus of this paper is to characterize fundamental tradeoffs between retail profits and consumer surplus, taking into account stochastic uncertainties in the whole sale market, the environmental (e.g. weather) factors, and end user operating conditions.

A. Related work

Among the earliest studies of dynamic pricing of electricity at the retail level are reported in [3], [5], in which the authors show that dynamic pricing can introduce economic incentives to the demand side and also improve the operation efficiency. The DADP scheme considered in this paper has the same structure as that in [3], [4], albeit the pricing scheme in these early papers is not optimized. In [3], DADP is shown to provide more benefits to consumers and can attract, in the long run, all the consumers,compared with other pricing schemes such as flat rate and TOU. The authors of [6], [7] present a more elaborate formulation of retail markets, which incorporates load models, retail profit functions, and financial risks. The idea of DADP was put in field trials. The authors of [4] conclude that such a pricing scheme "not only improves the linkage between wholesale and retail markets, but also promotes the development of retail competition." Relative to this line of existing work, the main contribution of this this paper is the optimized DADP and the use of a Stackelberg game model to obtain a full characterization of tradeoffs between the achievable retail profit and consumer surplus.

The idea of using Stackelberg game to study pricing of responsive load is considered in [8]. It is shown in [8] that the retailer can design a real-time retail price to lead the

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consumer to consume proper amount of electricity such that the social welfare maximization is achieved. In this paper, we also use the Stackelberg game model for load models that have significant statistical inter-temporal dependencies, and our objective goes beyond that of welfare maximization; we aim to characterize the Stackelberg equilibrium prices for a class of objective functions that can be used by the retailer. Recent work of Yang, Tang, and Nehorai [9] also viewed the interactive retail market structure as a game model. Their work is based on an abstract characterization of static interactions between a retailer and consumers.

There is an extensive line of work focusing on the optimization aspects of demand response, with pricing as a by-product of such optimizations. These approaches obtain optimized wholesale market price aimed at maximizing social welfare with the retailers (aggregators) acting as whole agents who can manage the consumers' energy consumption quantity directly. In [10], for a hierarchical market structure that includes the loads at the bottom layer, load service entities at the middle layer, and a independent service operator at the top, a distributed optimization algorithm is proposed with guaranteed convergence. The significance of this approach lies in that the optimization at the wholesale level (and the resulting wholesale pricing) does not require that the operator having full access to the consumer model at the bottom layer. The consumer preferences are represented iteratively by the load serving entities. In [11], a distributed optimization algorithm is proposed to solve the utility maximization at the retail level. Focusing on PHEV charging, the authors of [12] investigate the contract design among the renewable generators and the aggregator, who is responsible for the scheduling of the large scale PHEV charging.

B. Summary of results

Given that temperature control represents a substantial part of the energy consumption [13], we focus in this paper on thermal loads (HVACs units*) that can be controlled based on consumer preference and dynamic pricing. A challenging aspect of thermal load control is that the underlying thermal dynamics introduces inter-temporal correlation, making the problem of finding optimal response to dynamic pricing a stochastic dynamic program, which in general is not tractable.

For the consumer, we introduce the notion of consumer surplus (CS) as a linear combination of the energy cost and mean squared deviation from the preferred temperature. For the retailer, we define the retail profit (RP) as the revenue received from the consumer minus the retail cost associated with the real-time price in the wholesale market and loss in delivery. The social welfare is therefore the sum of consumer surplus and retail profit.

A main result of this paper is to provide a complete characterization of the Pareto front of the CS-RP trade-off, which is illustrated in Fig. 1. We show that the CS-RP plane is partitioned by a concave and monotonic decreasing Pareto front under which all CS-RP pairs can be achieved by a certain DADP, and any CS-RP pair above the Pareto front is not attainable by any DADP.

*Heating, ventilation, and air conditioning units

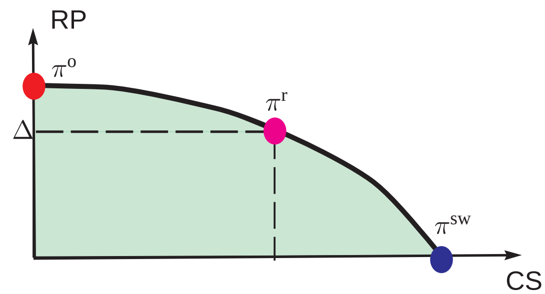

Fig. 1: CS-RP trade-off curve with three dynamic pricing schemes: social welfare maximizing price π^{sw}, regulated monopoly price π^{r}, and retail profit maximizing price π^{o}

We can place some well known pricing schemes on the CS-RP plane: the social welfare maximizing pricing π^{sw} is shown to be on the CS axis. The optimal regulated monopoly price π^{r} is located at the Pareto front where the retailer profit has a regulated profit margin Δ. The retailer profit maximizing price is π^{o} where no constraint is imposed on the retailer.

We also consider effects of incorporating renewable energy. It is showed that increased uncertainty of wind power will decrease both the retail profit and consumer surplus, hence shrink the Pareto front.

II. Stackelberg Game Model and Equilibria

The DADP scheme can be viewed as a Stackelberg game model [14]. The retailer acts as the leader who makes decision first, and the consumers act as the followers who observe the leader's action and then make the decision. Specifically, the action sequence in DADP is as follows:

- The retailer reveals the day ahead hourly price.
- The consumers schedule demands in real time.
- The retailer meets the demand subject to real time wholesale market price and integrates renewable resources.

The Stackelberg game can be solved via backward induction. We thus present first the analysis of demand side response to a fixed day ahead hourly price π. We then consider the problem of optimizing π.

A. Consumer action: optimal demand response

We consider in this section the optimization of the residential demand response to DADP. Assuming home energy management devices are implemented to optimize consumer surplus (utility minus cost), the resulting optimal demand response serves as the predicted behavior of a rational consumer in the Stackelberg game described above.

Let $\pi = (\pi_1, \cdots, \pi_{24})$ be the DADP vector where π_i is the day-ahead price for the ith hour. The home energy management system, on the other hand, typically operates in a much faster time scale, usually at the rate of 1 minute. Thus this is inherently a multi-time scale problem.

For the duration of 24 hours covered by the DADP, we assume that there are $N = 24 \times 60$ control periods. Let x_i be the average indoor temperature during period i. Empirical study [15] has shown that the dynamic equation that governs the temperature evolution is given by

$$x_i = x_{i-1} + \alpha(a_i - x_{i-1}) - \beta p_i + w_i, \tag{1}$$

where $a = (a_1, a_2, ..., a_N)$ is the vector of average outdoor temperature in each period, $p = (p_1, ..., p_N)$ the vector of control variable representing the total amount of electricity drawn by the HVAC unit during each period and $w = (w_1, w_2, ..., w_N)$ the process noise. System parameters α ($0 < \alpha < 1$) and β model the insolation of the building and efficiency of the HVAC unit. Note that the above equation applies to both heating and cooling scenarios but not simultaneously. We focus herein the cooling scenario ($\beta > 0$) and the results apply to heating ($\beta < 0$) as well.

To control the HVAC unit, temperature measurements need to be collected. We assume that thermometers are installed for both indoor and outdoor temperatures. The measurement equation is given by

$$y_i = \begin{bmatrix} x_i & a_i \end{bmatrix}^{\mathrm{T}} + v_i, \tag{2}$$

where v_i is the vector of measurement noise.

Assume that for minute i, the consumer wants to keep the indoor temperature close to the desired temperature t_i. The deviation of the actual indoor temperature x_i from t_i can be used to measure the consumer's uncomfort level. Hence, a reasonable consumer utility as a function of response action p is given by

$$u(\pi, \omega) = -\mu \sum_i (x_i - t_i)^2, \tag{3}$$

where μ is a weight factor to convert the temperature deviation to a monetary value. Note that the state variable x is affected by the response vector p.

Given the retail price π and the consumers' responsive demand p, the consumer surplus, $\mathsf{cs}(\pi, \omega)$ can be defined as the difference between the consumer utility (3) and total payment from the consumers to the retailer. Hereafter, we use ω to indicate random variables and vectors.

$$\mathsf{cs}(\pi, \omega) \stackrel{\Delta}{=} u(\pi, \omega) - \pi^{\mathrm{T}} U^{\mathrm{T}} p(\pi, \omega), \tag{4}$$

where U is the transform matrix to change hourly time scale to minute level time scale. With $\pi \in R^{24}$, $U\pi \in R^N$, $i.e.,$ $U\pi$ is a N-dimensional vector, and for all the periods in the same hour i, the prices all equal to π_i.

Under the DADP scheme, the consumption can be adjusted in real-time to maximize the expected consumer surplus. Specifically, the optimal stochastic residential demand response is defined as the solution to following optimal control problem,

$$\begin{array}{ll} \max\limits_{p} & \mathbb{E}\left\{ \sum_{i=1}^{N} [-\mu(x_i - t_i)^2] - \pi^{\mathrm{T}} U^{\mathrm{T}} p \right\} \\ \text{s.t.} & x_i = x_{i-1} + \alpha(a_i - x_{i-1}) - \beta p_i + w_i, \\ & y_i = (x_i, a_i) + v_i. \end{array}$$

where $y = (y_1, y_2, ..., y_N)$ is the observation vector, $v = (v_1, v_2, ..., v_N)$ the observation noise vector.

For computation convenience, under mild conditions (the price doesn't vary too much during a day and μ is large), we ignore the positive constraint and rate constraint for energy consumption p. This problem is similar to (but different from) the standard linear quadratic Gaussian (LQG) problem where the costs on the control and states are both quadratic. By backward induction, the following theorem [16] gives the optimal demand response.

Theorem 1: For fixed retail price π, the optimal aggregated residential demand response has the following matrix form and properties

$$p(\pi, \omega) = -GU\pi + b(\omega), \tag{5}$$

where the factor matrix G is positive definite and deterministic, depending only on the dynamic system parameters.

Proof: See Appendix. ■

Eq. (5) gives an affine form of residential demand response. The property that G is positive definite is important to our later discussion.

B. Retailer action: optimal DADP

In this paper, we assume that the retailer is a price taker in the wholesale market. This means that the retailer considered here serves customers whose aggregated demand does not affect the wholesale price in real-time. Additionally, we assume that the Stackelberg game discussed in this paper is with perfect information. It means that the form of the follower's payoff function is completely known to the leader.

The retailer needs to keep the balance of the power flow and provide electricity to the consumers. To achieve this, the retailer has to pay for the retail cost, including the distribution loss, the real-time payment to the wholesale market, and so on.

Let $\lambda(\omega) = (\lambda_1(\omega), \lambda_2(\omega), ..., \lambda_M(\omega))$ denote the random vector of per unit retail cost during each cost period (usually 5 minutes). In total, there are $M = 24 \times 12$ cost periods for a day. The retail profit, $\mathsf{rp}(\pi, \omega)$, is the difference between the real-time retail revenue and the retail cost, calculated as below,

$$\mathsf{rp}(\pi, \omega) = [U\pi - W\lambda(\omega)]^{\mathrm{T}} p(\pi, \omega), \tag{6}$$

where W are the transformation matrices to change $\lambda(\omega)$ to one minute based, similar to U in (4).

As the leader of the Stackelberg game, the retailer's pricing decision depends on its own payoff function. If the retailer only focuses on the expected retail profit, as in the classical monopoly case, and there is no correlation between real-time retail cost and demand, the solution to the following problem is the optimal pricing strategy.

$$\max_\pi \ \overline{\mathsf{rp}}(\pi) = (U\pi - W\bar{\lambda})^{\mathrm{T}}(-GU\pi + \bar{b}), \tag{7}$$

where bar is used to represent the expected value, also in the following part of this paper. As shown in Theorem 1, G is positive definite, hence the problem is a quadratic program.

However, as a load serving entity, the retailer needs to also take into consideration of consumer satisfaction measured by consumer surplus. Given the retail price π, by replacing the optimal demand response in Theorem 1 back into the consumer optimization problem, the expected consumer surplus can be expressed as

$$\begin{aligned} \overline{\mathsf{cs}}(\pi) &= \textstyle\sum_{i=1}^{N} [-\mu(x_i^* - t_i)^2] - \bar{p}^{*\mathrm{T}} U\pi \\ &= \pi^{\mathrm{T}} U^{\mathrm{T}} G U \pi / 2 - \pi^{\mathrm{T}} U^{\mathrm{T}} \bar{b}, \end{aligned} \tag{8}$$

where x^* and p^* are the same as the values in the proof of Theorem 1, see the Appendix. Thus, the expected consumer surplus is represented as a function of π.

The other extreme case is the retailer takes the expected social welfare, defined as the sum of consumer surplus and retail profit, as the payoff function. The expected social welfare, $\overline{\mathsf{sw}}(\pi)$ can be expressed as,

$$\overline{\mathsf{sw}}(\pi) = \overline{\mathsf{rp}}(\pi) + \overline{\mathsf{cs}}(\pi). \tag{9}$$

The social welfare reflects the combined benefit of the consumers and the retailer. By maximizing the expected social welfare, we can get the following theorem.

Theorem 2: The optimal retail price π^{sw} that maximizes the social welfare is given by the linear combination of the expected real-time retail cost, *i.e.*,

$$\pi^{\mathrm{sw}} = K\bar{\lambda},$$

where matrix K is a function of system parameters and user preferences, and $\overline{\mathsf{rp}}(\pi^{\mathrm{sw}}) = 0$. For any π' such that $\overline{\mathsf{rp}}(\pi') \geq 0$, we have $\overline{\mathsf{cs}}(\pi') \leq \overline{\mathsf{cs}}(\pi^{\mathrm{sw}})$.

Proof: See Appendix. ■

Theorem 2 shows that, if the social welfare is to be maximized, the retailer generates no profit. This result is consistent with the situation when there is perfect competition among identical retailers, in which case, social welfare maximization leads to zero profit.

A particularly informative case is when the time scale of demand response matches that of the hourly day ahead pricing. It can be shown in this case that $\pi^{\mathrm{sw}} = \mathbb{E}[\lambda(\omega)]$. In other words, in maximizing social welfare, the retailer simply matches the DADP with the expected real-time price, thus receiving no expected retail profit.

III. Achievable tradeoff

Now we extend the previous results to use a more general retail payoff function. Assume that the retailer's payoff function is a linear combination of retail profit and consumer surplus as follows

$$\max \ \{\overline{\mathsf{rp}}(\pi) + \eta\overline{\mathsf{cs}}(\pi)\}, \tag{10}$$

where η is the preference of the retailer on the consumer surplus. If $\eta = 1$, this is equivalent to optimizing the social welfare. If $\eta = 0$, this is equivalent to optimizing the retail profit. Therefore, a rational retailer should choose η between $[0, 1]$, depending on how much the consumer surplus is considered. The η's beyond this region doesn't make sense since that $\eta < 0$ means the retailer can benefit from reducing the consumer's surplus, and $\eta > 1$ means that the retailer will reduce its own profit (maybe to negative) to achieve better payoff at the maximized social welfare, which is usually irrational for the retailer.

An alternative formulation is the optimal CS-RP trade-offs. In particular, we are interested in characterizing the Pareto front involving $(\overline{\mathsf{cs}}(\pi), \overline{\mathsf{rp}}(\pi))$.

A point on the Pareto front can be obtained by considering a practical situation where the retailer optimizes its profit under the constraint that the consumer surplus exceeds a certain level. In particular,

$$\begin{aligned} \max \quad & \overline{\mathsf{rp}}(\pi) \\ \text{s.t.} \quad & \overline{\mathsf{cs}}(\pi) \geq \tau. \end{aligned} \tag{11}$$

From the above optimization, the Pareto front can be traced by varying the consumer surplus level, as shown in Fig 1.

Theorem 3: For any specific η, if the solution in (10) is π^*, π^* is also a solution to (11) with $\tau = \overline{\mathsf{cs}}(\pi^*)$. Varying τ in optimization (11) and varying η in optimization (10) will give the same trade-off curve between expected retail profit and expected consumer surplus.

Proof: See Appendix. ■

Theorem 3 implies that the two optimization problems are equivalent and give the same Pareto front (Fig. 1). Each point on the Pareto front is attainable and corresponds to a equilibrium point in the Stackelberg game with particular payoff function. The following theorem shows the concave and optimality property of this trade-off curve. The retailer's pricing capability is constrained by this trade-off curve.

Theorem 4: The Pareto front of $(\overline{\mathsf{cs}}, \overline{\mathsf{rp}})$ is concave and decreasing.

Proof: See Appendix. ■

We can place some well known pricing strategies on the Pareto front, as shown in Fig.1. The social welfare maximaxing pricing π^{sw} is located on the CS axis. This is intuitive since maximizing social welfare dictates the removal of retail profit. The optimal regulated monopoly price π^{r} is located at the Pareto front where the retailer profit has a regulated profit margin, Δ (see Fig.1). π^{o} is the price when the retailer's objective is purely maximizing its profit.

IV. Benchmark Comparisons

In this section, we compare some benchmark schemes and place them in the achievable CS-RP tradeoff region. The following two benchmark pricing schemes are considered:

- Constant pricing: in this case, the price remains constant for the whole day, $i.e.,\ \pi_1 = \pi_2 = ... = \pi_{24}$. By varying the price value, we get the CS-RP pairs.
- Constant mark-up pricing: in this case, the ratio of day-ahead price to the expected real-time price remains the

same, $i.e.$, $\frac{\pi_1}{\mathbb{E}\lambda_1} = \frac{\pi_2}{\mathbb{E}\lambda_2} = ... = \frac{\pi_{24}}{\mathbb{E}\lambda_{24}}$. Similarly, by varying the ratio, the CS-RP trade-off is plotted.

We used the actual temperature record in Hartford, CT, from July 1st, 2012 to July 30th, 2012. The day-head price (used as prediction) and real-time price (used as realization) are also for the same period from ISO New England. The HVAC parameters for the simulation is set as: $\alpha = 0.5$, $\beta = 1$, $\mu = 10$. The desired indoor temperature is set to be $18°C$ for all hours.

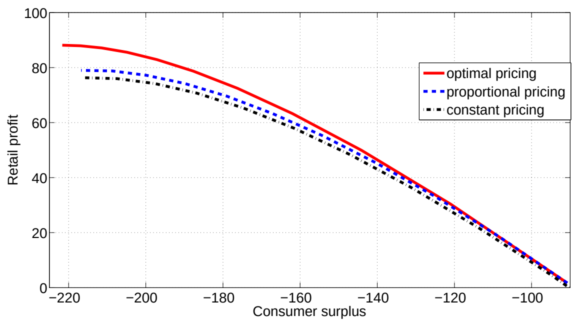

Fig. 2: comparasion of three pricing schemes

The result in Fig. 2 shows that the trade-off curve corresponding to the first two pricing schemes both fall below the Pareto front as we expected. Indeed, the optimal tradeoff curve is the Pareto front for the retailer, and CS-RP pairs will stay below the front.

V. Effect of renewable energy

As a large load aggregator, the retailer has the ability to build its own facility. We now consider a scenario that the retailer has a large wind farm, the power from which can also be used to compensate the real-time load from the consumers. Technically, serving the local area is much easier than serving back the grid, which may require additional devices and may cause instability. Hence, we restrict ourselves to the case that the retailer cannot sell the wind power back to the wholesale market. The reason for this is that the retailer we model here is not participating the day-ahead market schedule thus sending power back to grid will increase the instability of the system which is beyond the scope of this paper.

Assume that the marginal cost of wind power is zero. Denote the random wind power as $q(\omega) = (q_1(\omega), ..., q_N(\omega))$. Notice here we assume the wind power is in the same scale as the residential demand. Then, the retailer's profit is changed to

$$\begin{aligned} \mathsf{rp}_{\mathrm{w}}(\pi, \omega) = \quad & \pi^{\mathrm{T}} U^{\mathrm{T}} [-GU\pi + b(\omega)] \\ & -\lambda(\omega)^{\mathrm{T}} W^{\mathrm{T}} [-GU\pi + b(\omega) - q(\omega)]^{+}, \end{aligned}$$

where the function $(x)^+$ is the positive part of x, defined as $\max\{x, 0\}$.

Notice that the expected retail profit is also a concave function of π. Thus the profit maximization problem can be solved by a convex program. With a fixed π, the retail profit is a increasing function of q, and the consumer surplus is independent of q. Comparing with the case without wind power, $i.e.$, $q = 0$, the Pareto front is enlarged with the existence of positive wind power.

In evaluating effects of wind power on the consumer surplus-retail profit tradeoff, we conducted numerical simulations. The parameter setting is the same as Section IV. Assume the wind power is free to the retailer and uniformly distributed over $[\bar{q}-\Delta, \bar{q}+\Delta]$, where the $\bar{q}$ is the mean value. Here we compare four cases: no wind exists, constant wind, wind with low uncertainty, and wind with high uncertainty. For the latter three, we mean fixing $\bar{q}$ but increasing Δ in order.

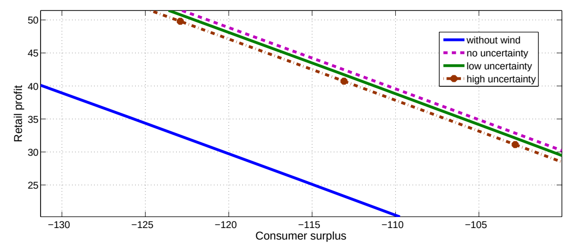

Fig. 3: Effect of incorporating wind

Figure 3 shows that with wind power, the area under the Pareto front is enlarged comparing with the case without wind. On the other hand by fixing $\bar{q}$ and increasing Δ, the area under Pareto front shrinks. This means making a better wind prediction will benefits the retailer's pricing behavior.

VI. Conclusion

We have studied in this paper the DADP—a day ahead dynamic pricing mechanism for demand response in uncertain and dynamic environments. Such a pricing scheme has the advantage of reducing consumer anxiety of pricing uncertainties and allowing the retailer optimize the retail pricing to protect itself from uncertainties in the wholesale market and uncertainties existed in the consumer end.

We formulate the problem as a Stackelberg game in which the retailer plays the role of a leader and the consumers the follower. For thermal dynamic load and price inelastic random load, we obtain the optimal demand response to DADP from which the retailer optimizes payoff functions of its own choice.

The DADP framework established in this paper allows us to study a number of benchmark pricing mechanisms and place them in the achievable region of performance tradeoff between consumer surplus and retail profit. It also provided insights into the role of renewable sources procured by the retailer.

Appendix

Proof: (Theorem 1) Solving the optimal dynamic program by backward induction, the optimal strategy is

$$
\begin{array}{rcl}
p_i^* & = & \frac{1}{\beta}\left(\hat{x}_{i-1|i-1}+\alpha(\hat{a}_{i|i-1}-\hat{x}_{i-1|i-1})-x_i^*\right),\\
x_i^* & \triangleq & \frac{U_i\pi-(1-\alpha)U_{i+1}\pi}{2\mu\beta}+t_i,
\end{array}
$$

where $\hat{x}_{i-1|i-1}$ and $\hat{a}_{i|i-1}$ are the estimated indoor and outdoor temperatures based on observations up to period $i-1$ respectively, U_i represents the ith row of U, and x_i^* is an ancillary value.

Let superscript k denote the parameter associated with consumer k. Expanding the solution above will give $p_i^{(k)}(\pi,\omega)=\frac{1}{\beta^{(k)}}\{(1-\alpha^{(k)})U_{i-1}\pi-[1+(1-\alpha^{(k)})^2]U_i\pi+(1-\alpha^{(k)})U_{i+1}\pi\}+b^{(k)}(\omega)$, where $b^{(k)}(\omega)$ is independent on π. So the total demand of consumer k is $p^{(k)}(\pi,\omega)=-G^{(k)}U\pi+b^{(k)}(\omega)$, where $G^{(k)}$ satisfies

$$
G_{ij}^{(k)}=\begin{cases}[1+(1-\alpha^{(k)})^2]/\beta^{(k)} & \text{if } i=j\\ (-1+\alpha^{(k)})/\beta^{(k)} & \text{if } |i-j|=1\\ 0 & \text{o.w.}\end{cases}
$$

Notice that $G^{(k)}$ is deterministic and diagonal dominant with positive diagonal elements. Hence, $G^{(k)}$ is positive definite.

On the other hand, the aggregated demand

$$
p(\pi,\omega)=\sum_k p^{(k)}(\pi,\omega)=-GU\pi+b(\omega),
$$

where $b(\omega)=\sum b^{(k)}(\omega)$, $G=\sum G^{(k)}$. Since $G^{(k)}$ is positive definite and deterministic, G is also positive definite and deterministic, depending only on the dynamic system parameters. ■

Proof: (Theorem 2) Since G is positive definite, the formula of expected social welfare, $\overline{\mathsf{sw}}(\pi)$, is a concave function of π. So taking derivative and setting it to zero will give us the optimal price for social welfare maximization as below,

$$
\pi^{\text{sw}}=(U^TGU)^{-1}U^TGW\bar{\lambda}.
$$

On the other hand, replacing π^{sw} will make the retail profit

$$
\begin{array}{rl}
\overline{\mathsf{rp}}(\pi^{\text{sw}}) & =[U(U^TGU)^{-1}U^TGW\bar{\lambda}-W\bar{\lambda}]^T\\
& \cdot[-GU(U^TGU)^{-1}U^TGW\bar{\lambda}+\bar{b}]\\
& =0.
\end{array}
$$

■

Proof: (Theorem 3) With a particular η, assume π^* is a solution to (10). Let $\tau=\overline{\mathsf{cs}}(\pi^*)$ in (11). Then π^* will be in the feasible set of (10). If there exists π', such that $\overline{\mathsf{rp}}(\pi')>\overline{\mathsf{rp}}(\pi^*)$, and $\overline{\mathsf{cs}}(\pi')\geq\tau$, then

$$
\begin{array}{rl}
\overline{\mathsf{rp}}(\pi')+\eta\overline{\mathsf{cs}}(\pi') & >\overline{\mathsf{rp}}(\pi^*)+\tau\\
& \geq\overline{\mathsf{rp}}(\pi^*)+\eta\overline{\mathsf{cs}}(\pi^*).
\end{array}
$$

Hence, π^* is not the solution to (10) since π' achieves better objective value. It contradicts with the assumption. Therefore, π^* is also a solution to (11). ■

Proof: (Theorem 4) For $\eta\in[0,1]$, the optimization of (10) can be expressed as

$$
\begin{array}{ll}
\max & (\pi^{\mathrm{T}}U^{\mathrm{T}}-\bar{\lambda}^{\mathrm{T}}W^{\mathrm{T}})(-GU\pi+\bar{b})\\
& +\eta(\pi^{\mathrm{T}}U^{\mathrm{T}}GU\pi/2-\pi^{\mathrm{T}}U^{\mathrm{T}}\bar{b}).
\end{array}
$$

Hence, the solution is given by

$$
\pi^*=\tfrac{1}{2-\eta}(U^{\mathrm{T}}GU)^{-1}[(1-\eta)U^T\bar{b}+U^TGW\bar{\lambda}].
$$

Define the resulted retail profit and consumer surplus as

$$
\begin{array}{c}
\mathsf{rp}^*(\eta)\triangleq\overline{\mathsf{rp}}(\pi^*(\eta));\\
\mathsf{cs}^*(\eta)\triangleq\overline{\mathsf{cs}}(\pi^*(\eta)).
\end{array}
$$

It can be showed that,

$$
\frac{\partial\mathsf{rp}^*(\eta)}{\partial\mathsf{cs}^*(\eta)}=\frac{\frac{\partial\mathsf{rp}^*(\eta)}{\partial\eta}}{\frac{\partial\mathsf{cs}^*(\eta)}{\partial\eta}}=-\eta.
$$

On the other hand, it can be verified that $\mathsf{cs}^*(\eta)$ is a increasing function of η. Therefore, $\frac{\partial\mathsf{rp}^*(\eta)}{\partial\mathsf{cs}^*(\eta)}$ decreases as $\mathsf{cs}^*(\eta)$ increases. The curve is concave. ■

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