ERROR EXPONENTS FOR BAYESIAN DETECTION WITH RANDOMLY SPACED SENSORS

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ABSTRACT

We study the detection of Gauss-Markov signals using randomly spaced sensors. We derive a lower bound on the Bayesian detection error based on the Kullback-Leibler divergence, and from this, define an error exponent. We then evaluate the error exponent for stationary and non-stationary Gauss-Markov models where the sensor spacings, $d_1, d_2, \ldots$, are drawn independently from a common distribution F_d. In both models, error exponents take on simple forms involving the parameters of the Markov process and expectations over F_d of certain functions of d_1. These expressions are evaluated explicitly when F_d corresponds to (i) exponentially distributed sensors with placement density λ (ii) equally spaced sensors, and (iii) the proceeding cases when sensors also fail with probability q. Many insights follow. For example, in the non-stationary case, we determine the optimal λ as a function of q. Numerical simulations show that the error exponent, based on an asymptotic analysis of the lower bound, predicts trends of the actual error rate accurately, even for small data sizes.

***Index Terms*—** Distributed Bayesian detection, Error exponent, Gauss-Markov, Sensors, Optimal placement density

1. INTRODUCTION

We study the detection of a signal by a sensor network as depicted in Figure 1. We assume that the signal is present under two hypotheses, $\mathcal{H}_j, j \in \{0, 1\}$, and that it has a Gauss-Markov correlation structure and power level that are hypothesis-dependent. The signal field is sampled by a set of N sensors, and these samples are delivered to a fusion center (FC). The FC then makes a single global decision as to the true hypothesis using Bayesian hypothesis testing [11]. We assume that the sensors are randomly placed along a straight line.[1] This is because sensors are often deployed without precise control (e.g., they are air dropped in military applications). Further, even if sensors are equally spaced upon deployment, sensor failures introduce randomness into the spacing between operational sensors. We study the theoretical detection performance once the N samples arrive at the FC.[2] As an example in which this model is relevant, consider sensors deployed ad-hoc in a hostile environment and tasked with detecting the class of a tank, either friendly or enemy, based on the acoustic wavefront that it produces. This wavefront is a signal field that can be sampled by acoustic sensors. The power and correlation structure of these samples would depend on the tank's class as well as the (random) locations of sensors.

[1] The straight line assumption is for clarity of the exposition. Section 5 discusses the extension to more general placements.

[2] The communication protocols used to initiate the detection process and to deliver the samples to the FC are not considered in this work.

1.1. Notation

We use the following notation and definitions: (a) $\mathbb{E}(.)$ denotes expectation. When there is potential ambiguity, $\mathbb{E}_X(.)$ denotes expectation with respect to (w.r.t.) a random variable X, while $\mathbb{E}_{\mathcal{H}_j}(.)$ denotes expectation w.r.t. the hypothesis $\mathcal{H}_j$, (b) $\mathcal{N}(0, \sigma^2)$ denotes a zero mean Gaussian random variable with variance σ^2, (c) if $f(x) = O(x^k)$, then $|f(x)| < Cx^k$ for some $C \in \mathbb{R}^+$ and all sufficiently small x, (d) boldface lowercase letters, e.g. $\mathbf{x}$, denote vectors, (e) x_1^N denotes the sequence $x_1, \ldots, x_N$, and (f) $p_j(\mathbf{x})$ denotes the probability density of $\mathbf{x}$ under $\mathcal{H}_j$.

1.2. Background

Consider a general binary hypothesis test between $\mathcal{H}_0$ and $\mathcal{H}_1$. Let $\mathbf{y} = [y_1, \ldots, y_N]^T \in \mathbb{R}^N$ be a vector of observed data. Let π_j be the prior probability of $\mathcal{H}_j$. The Bayesian error is defined

$$P_e \triangleq \pi_0 \; P(\text{ error} \mid \mathcal{H}_0) + \pi_1 \; P(\text{ error} \mid \mathcal{H}_1),$$

where $P(\text{ error} \mid \mathcal{H}_j)$ is the detection error when $\mathcal{H}_j$ is true, and where we assume that the maximum a posteriori probability (MAP) detector is used. Without loss of generality, let the function $K(N)$ be defined so that $P_e = e^{-N\,K(N)}$ for all N. The quantity of interest is the exponential rate of decay in P_e as the number of signal samples approaches infinity, i.e.,

$$K \triangleq \lim_{N\to\infty} K(N), \tag{1}$$

provided that the limit exists and is independent of N. Exact, implicit, expressions for K are available [2]. However, these expressions are rarely tractable for specific signal models. Thus, more tractable *approximate* methods are often used to characterize detection performance.

A common approach is define an error exponent based on an upper bound to P_e. For example, it is well known that $P_e \le e^{-NK_U(N)}$, for all N, where $K_U(N)$ is such that

$$K_U \triangleq \lim_{N\to\infty} K_U(N) = -\lim_{N\to\infty} \frac{1}{N} \min_{s\in[0,1]} \ln \mathbb{E}_{\mathcal{H}_0} \left\{ \left(\frac{p_1(\mathbf{y})}{p_0(\mathbf{y})} \right)^s \right\}, \tag{2}$$

[11, p.89]. In the special case that $\{y_k\}$ are independent and identically distributed (i.i.d.), $K_U = K$ (e.g., see [3]). Unfortunately, K_U is intractable for the (non-i.i.d.) models we study in this paper.

To obtain tractability, we lower bound P_e instead. From this, we are led to an error exponent that is seen to be the normalized limit of the Kullback-Leibler divergence between $p_0(.)$ and $p_1(.)$. Specifically, we first show that $P_e \ge e^{-NK_L(N)}$ for all N, where $K_L(N)$ is given in Section 2. It will be seen that

$$K_L \triangleq \lim_{N\to\infty} K_L(N) = \lim_{N\to\infty} \frac{1}{N} \mathbb{E}_{\mathcal{H}_0} \left(\ln \frac{p_0(\mathbf{y})}{p_1(\mathbf{y})} \right). \tag{3}$$

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We then evaluate K_L explicitly for the signal models of interest. It will be seen that K_L remains tractable and permits a thorough analysis of the effect of sensor placement density and failure rate on the error exponent.

1.3. Organization and Main Results

In Section 2, we derive $K_L(N)$ in (4), and show that it leads to the error exponent K_L in (5). We then study K_L for Gauss-Markov signals under both hypotheses when sensor spacings are drawn i.i.d. from an arbitrary distribution function F_d, for both the stationary (7) and non-stationary (16) cases. We consider the non-stationary case in Section 3. The error exponent is seen to simplify to closed form expressions under the following (physically motivated) choices for F_d: (i) exponentially distributed sensors with placement density λ (ii) equally spaced sensors with spacing $\bar{d}$, and (iii) the proceeding cases when sensors fail with probability q, e.g., see (10) and (12). For exponentially distributed sensors with failures, the optimal sensor placement density is found in (14). We consider the stationary case in Section 4. We evaluate the error exponent in closed form, in terms of the Psi function, when F_d corresponds to exponentially distributed sensors with failures, see (17). This expression is seen to simplify in the limit of sparsely and densely placed sensors in (18) and (19). Numerical simulations of the true exponential rate of error decay are used throughout to show that the analytic framework presented here allows for an accurate and efficient optimization of system resources; one that would not be possible otherwise.

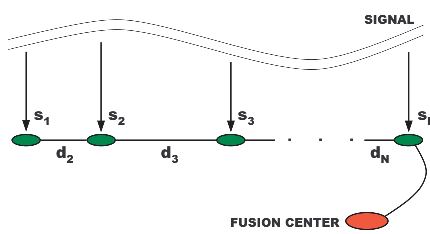


Fig. 1. The collection of N signal samples from randomly placed sensors. $\{s_k\}_{k=1}^N$ denotes the collected samples, and $\{d_k\}_{k=2}^N$ denotes the random sensor spacings. The data $\{s_k\}$ and $\{d_k\}$ are observed and sent to a fusion center, which makes a global decision as to the true hypothesis.

1.4. Related Work

Error exponents of the type (1)-(3) have been used to study a wide range of issues in distributed detection, including sensor-data quantization schemes, the number of sensors to transmit to a FC, and the placement of sensors in hierarchical networks.

In the Bayesian setup, error exponents based on (2) or the similar Bhattacharya bound can be found in [6], [4], and [12], among others. In [6], the Chernoff and Bhattacharya bounds are used to determine the optimal number of sensors for communications to a FC under power and bandwidth constraints. In [4], an error exponent is provided when sensors have dependent observations, and the optimal sensor density is found when sensors are equally spaced. In [12], a routing scheme is designed based on the Chernoff information when the sensor placement is deterministic.

In the Neyman-Pearson setup, [4] provides an error exponent for dependent data with equally spaced sensors, [13] derives an error exponent when the signal is a stationary, Gauss-Markov signal under one hypothesis and i.i.d. noise under the other. In [1, pp.138-139] and [9], error exponents are provided for Gauss-Markov signals under both hypotheses. In [10], we previously characterized properties of the error exponent for Gauss-Markov signals under both hypotheses, using a physical model which linked the correlation parameter to network design parameters. However, none of the works discussed thus far provide insights for the case when sensors are randomly deployed, motivating the need for the analysis presented in this paper.

2. BOUND ON BAYESIAN ERROR

We develop a lower bound on Bayesian error of binary hypothesis testing. Letting $N \to \infty$ in this bound, we define K_L. The expressions derived here will be evaluated explicitly in later sections.

We can show that

$$\begin{aligned} P_e &\overset{(a)}{\geq} \pi_0\pi_1 \left[\mathbb{E}_{\mathcal{H}_0}\left(\sqrt{\frac{p_1(\mathbf{y})}{p_0(\mathbf{y})}}\right)\right]^2 \\ &\overset{(b)}{=} \pi_0\pi_1 \exp\left\{2\ln\left[\mathbb{E}_{\mathcal{H}_0}\left(e^{\frac{1}{2}\ln\frac{p_1(\mathbf{y})}{p_0(\mathbf{y})}}\right)\right]\right\} \\ &\overset{(c)}{\geq} \pi_0\pi_1 \exp\left\{\mathbb{E}_{\mathcal{H}_0}\left[\ln\frac{p_1(\mathbf{y})}{p_0(\mathbf{y})}\right]\right\} \\ &\overset{(d)}{=} e^{-N\left[\mathbb{E}_{\mathcal{H}_0}\left(\frac{1}{N}\ln\frac{p_0(\mathbf{y})}{p_1(\mathbf{y})}\right)-\frac{1}{N}\ln\pi_0\pi_1\right]} \end{aligned} \tag{4}$$

where (a) is a general bound shown in [8] (see also [11, p.90]), (b) and (d) follow from algebraic manipulation, and (c) follows from Jensen's inequality [7, p.249]. It can be shown that the RHS of (4) is $\in (0, 1/2]$, and so the bound is non-trivial. The term in brackets in (4),

$$K_L(N) \triangleq \frac{1}{N}\mathbb{E}_{\mathcal{H}_0}\left(\ln\frac{p_0(\mathbf{y})}{p_1(\mathbf{y})}\right) - \frac{1}{N}\ln\pi_0\pi_1,$$

describes the decay of the error rate as a function of N. Observe that

$$K_L \triangleq \lim_{N\to\infty} K_L(N) = \lim_{N\to\infty}\frac{1}{N}\mathbb{E}_{\mathcal{H}_0}\left(\ln\frac{p_0(\mathbf{y})}{p_1(\mathbf{y})}\right). \tag{5}$$

In the remainder of this paper we will concentrate on evaluating the error exponent K_L explicitly for certain signal models.

3. NON-STATIONARY MARKOV MODEL

We model the signal under each hypothesis as a Gaussian signal that evolves with a Gauss-Markov correlation structure along any straight line. Consider the observations s_1^N taken by the sensors. We assume that s_1^N are noiseless, with statistics under $\mathcal{H}_j$ described by

$$\begin{aligned} &s_1 \sim \mathcal{N}(0,\ \sigma_{j,I}^2), \\ &s_k = a_{j,k}\ s_{k-1} + z_{j,k}, \quad k \geq 2, \end{aligned}$$

where $a_{j,k} \in (0,1)$ describes the correlation strength between the $(k-1)$th and kth sensors, and $z_{j,k} \overset{i.i.d.}{\sim} \mathcal{N}(0,\ \sigma_j^2)$ is impulsive (or innovations) noise. Let $\{d_k\}$ be the i.i.d. sequence of sensor spacings. We assume $d_k \sim F_d'$, independent of $\mathcal{H}_j$, and that $a_{j,k}$ is a function of d_k, i.e.,

$$a_{j,k} = g_j(d_k),$$

where $g_j(.)$ is a hypothesis-dependent deterministic function (for an example, see (8)). We assume that $\{s_k\}_{k=1}^N$ and $\{d_k\}_{k=2}^N$ are delivered to the FC. Finally, it will be convenient to define $d \triangleq d_k$ and $a_j \triangleq g_j(d)$ for use in expressions where the index k is irrelevant.

3.1. Derivation of the Error Exponent

Note that $\mathbf{y} = [\mathbf{d}, \mathbf{s}]$ summarizes the data delivered to the FC, where $\mathbf{d} \triangleq [d_2, \ldots, d_N]$ and $\mathbf{s} \triangleq [s_1, \ldots, s_N]$. Evaluating the argument of the expectation in (5), we get

$$\ln \frac{p_0(\mathbf{s}, \mathbf{d})}{p_1(\mathbf{s}, \mathbf{d})} \stackrel{(a)}{=} \ln \frac{p_0(\mathbf{s} \,|\, \mathbf{d})}{p_1(\mathbf{s} \,|\, \mathbf{d})} \stackrel{(b)}{=} \ln \frac{p_0(s_1)}{p_1(s_1)} \prod_{k=2}^{N} \frac{p_0(s_k | s_{k-1}, d_k)}{p_1(s_k | s_{k-1}, d_k)}$$
$$\stackrel{(c)}{=} \frac{1}{2} \ln \frac{\sigma_{1,I}^2}{\sigma_{0,I}^2} + s_1^2 \frac{\sigma_{0,I}^2 - \sigma_{1,I}^2}{2\sigma_{0,I}^2 \sigma_{1,I}^2} + \frac{N-1}{2} \ln \frac{\sigma_1^2}{\sigma_0^2}$$
$$+ \sum_{k=2}^{N} \left\{ \frac{(s_k - a_{1,k} s_{k-1})^2}{2\sigma_1^2} - \frac{(s_k - a_{0,k} s_{k-1})^2}{2\sigma_0^2} \right\}, \qquad (6)$$

where (a) follows since $p_j(\mathbf{d})$ is independent of j, (b) follows since $\mathbf{s}$ is a Markovian process given $\mathbf{d}$, and (c) follows from the form of the conditional Gaussian distribution. Next, we have

$$\mathbb{E}_{\mathcal{H}_0} \left(\ln \frac{p_0(\mathbf{s}, \mathbf{d})}{p_1(\mathbf{s}, \mathbf{d})} \right) = \mathbb{E}_{\mathcal{H}_0} \left(\mathbb{E}_{\mathcal{H}_0} \left(\ln \frac{p_0(\mathbf{s}, \mathbf{d})}{p_1(\mathbf{s}, \mathbf{d})} \,|\, \mathbf{d} \right) \right)$$
$$= \frac{1}{2} \ln \frac{\sigma_{1,I}^2}{\sigma_{0,I}^2} + \frac{1}{2} \left[\frac{\sigma_{0,I}^2}{\sigma_{1,I}^2} - 1 \right] + \frac{N-1}{2} \left(\ln \frac{\sigma_1^2}{\sigma_0^2} - 1 \right)$$
$$+ \frac{(N-1)\sigma_0^2}{2\sigma_1^2} + \frac{\mathbb{E}_d \left[(a_0 - a_1)^2 \right]}{2\sigma_1^2} \left(\sigma_{0,I}^2 \frac{1 - \mathbb{E}_d \left[a_0^2 \right]^{N-1}}{1 - \mathbb{E}_d \left[a_0^2 \right]} \right.$$
$$\left. + \sigma_0^2 \frac{\mathbb{E}_d \left[a_0^2 \right]^{N-1} + (N-2) - \mathbb{E}_d \left[a_0^2 \right] (N-1)}{(1 - \mathbb{E}_d \left[a_0^2 \right])^2} \right),$$

where we have omitted the lengthy calculations. Taking the limit as $N \to \infty$, we get the error exponent to be

$$K_L = \frac{1}{2} \left\{ \ln R - 1 + \frac{1}{R} \left\{ 1 + \frac{\mathbb{E}_d \left[(g_0(d) - g_1(d))^2 \right]}{1 - \mathbb{E}_d \left[g_0(d)^2 \right]} \right\} \right\}, \quad (7)$$

where $R \triangleq \sigma_1^2 / \sigma_0^2$. Equation (7) is valid when F_d is a continuous or discrete distribution (examples of each are given in Section 3.1.1). If the sensor spacing is deterministic, the expectations above disappear, and (7) reduces to the error exponent given in [9] for Neyman-Pearson detection.

3.1.1. Example 1

We now evaluate (7) for several models of the sensor spacing. In each, we assume that $a_{j,k}$ decays exponentially in d_k at a rate proportional to a constant A_j, i.e.,

$$g_j(x) = e^{-A_j x}, \qquad (8)$$

for $x \geq 0$, where $A_j \in (0, \infty)$ is known at the FC, and $A_0 \neq A_1$.

Example 1(a): Exponentially spaced sensors

Let the sensor spacings be exponentially distributed with placement density (or "arrival rate") λ . We have

$$F_d'(x) = \lambda\, e^{-\lambda x}, \; x \geq 0, \qquad (9)$$

where $\lambda = 1/\mathbb{E}(d)$, by property of the exponential. Evaluating (7) with (8) and (9), we get

$$K_L = \frac{1}{2} \left\{ \ln R - 1 + \frac{1}{R} \left(1 + \frac{(A_1 - A_0)^2 \lambda}{A_0 (A_0 + A_1 + \lambda)(2A_1 + \lambda)} \right) \right\}. \qquad (10)$$

It can be verified that (10) is unimodal in λ and is maximized when

$$\lambda = \lambda^\star \triangleq \sqrt{2A_1(A_0 + A_1)},$$

i.e., there exists an optimal placement density. Larger A_0 and/or A_1 implies that a higher placement density is optimal, while smaller A_0 and/or A_1 implies that a lower placement density is optimal.

In Figure 2 we plot K_L and $K(N)$ with $N = 20$ (determined numerically) versus λ for $A_1 = 1/10$ and $A_0 \in \{1/4, 1/2, 1, 2, 5\}$ (other parameters are given in the caption). It is seen that the behavior predicted by K_L holds for $K(N)$. For example, $K(N)$ is seen to be unimodal in λ for each A_0. Further, the optimal placement density predicted by K_L is seen to hold for $K(N)$. For example, when $A_0 = 1/2$ we get $\lambda^* = 0.34\ldots$, while $K(N)$ is maximized for $\lambda = 0.36\ldots$. While K_L and $K(N)$ have similar behavior versus λ, the magnitude of K_L is larger than for $K(N)$. While the analytic framework allows for an accurate and efficient optimization of system resources in this example, the convergence of the error exponent in magnitude may be slow in N.

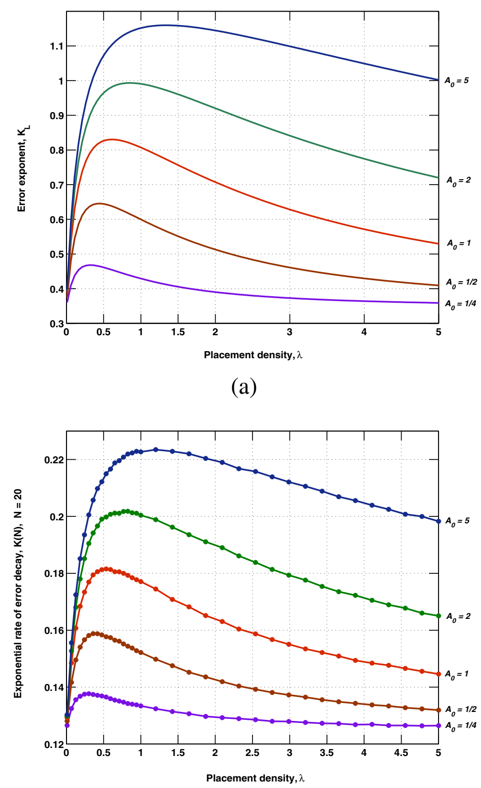


Fig. 2. Bayesian detection performance versus the placement density λ when $A_1 = 1/10$ and $A_0 \in \{1/4, 1/2, 1, 2, 5\}$ for (a) the error exponent K_L, and (b) the true error rate $K(N)$ with $N = 20$. Parameters: $\pi_0 = \pi_1 = \sigma_{0,I}^2 = \sigma_{1,I}^2 = 0.5$, $\sigma_0^2 = 2.5$, and $\sigma_1^2 = 1$.

Example 1(b): Equispaced sensors with failures

Next, suppose sensors are in failure with probability q independently from sensor to sensor. A "failed" sensor is a sensor whose data is not received at the FC. Typical reasons for failure may include mechanical malfunction or battery depletion at the sensor, and lost transmissions due to interference at the FC. In addition, by choosing

q appropriately, the analysis below incorporates probabilistic transmission schemes (in which a node transmits its data only with some probability; such a scheme was shown to provide the optimal trade-off between error exponents and energy consumption in [14]) and schemes in which sensors enter cycled "sleep" states.

Let $\bar{d}$ be the spacing of sensors upon deployment and let $q \in [0, 1)$ be the failure rate. Then

$$F_d'(x) = \begin{cases} (1-q)q^{n-1}, & x = n\bar{d} \\ 0, & x \neq n\bar{d} \end{cases}, \qquad (11)$$

for $n \in \{1, 2, \ldots\}$, is the probability density of the spacing between operational sensors. The error exponent is evaluated using (7), (8), and now (11). We get

$$K_L = \frac{1}{2}\Bigg\{ \ln R - 1 + \frac{1}{R}\left(1 + \frac{\left(e^{A_0\bar{d}} - e^{A_1\bar{d}}\right)^2 (1-q)(e^{(A_0+A_1)\bar{d}} + q)}{(e^{2A_0\bar{d}} - 1)(q - e^{2A_1\bar{d}})(q - e^{(A_0+A_1)\bar{d}})}\right)\Bigg\}. \qquad (12)$$

Unfortunately, a general closed form expression for the optimal $\bar{d}$ as a function of q is not available, but see the comments below.

Theoretical curves generated from (12) are shown in Figure 3(a), where we plot K_L versus $\bar{d}$ for $q \in \{0.0, 0.1, 0.3, 0.5, 0.7\}$ (other parameters are given in the caption). It is seen that, for each value of q, the error exponent is unimodal in $\bar{d}$ with an optimal $\bar{d}$ that decreases in q. When $\bar{d} \ll 1$, a Taylor series analysis of (12) reveals that K_L is increasing in both $\bar{d}$ and q, and when $\bar{d} \gg 1$, that K_L is decreasing in $\bar{d}$ and q. In Figure 3(b), we plot $K(N)$ when $N = 12$ for the same parameters. The numerical curves coincide with theoretical predictions based on K_L: $K(N)$ is seen to be unimodal in $\bar{d}$ with an optimal spacing that decreases with q. When $\bar{d} \ll 1$, $K(N)$ appears increasing in both $\bar{d}$ and q, and when $\bar{d} \gg 1$, $K(N)$ appears decreasing in $\bar{d}$ and q. Also note that the magnitude of K_L and $K(N)$ are in relatively close agreement. We conclude that K_L predicts the behavior of $K(N)$, even for small samples sizes.

Example 1(c): Exponentially spaced sensors with failures

Consider the case where sensors spacings are exponentially distributed with placement density λ and with failure probability q. In this case, $d_k = \sum_{\ell=1}^{M} \widetilde{d}_{k,\ell}$, where $\{\widetilde{d}_{k,\ell}\}_\ell$ is an i.i.d. sequence with common probability density given by the RHS of (9), and where $P[M = k] = q^{k-1}(1-q)$ for $k \in \{1, 2, ..\}$. It can be verified that the probability density of the spacing between two consecutive operational sensors is

$$F_d'(x) = \lambda(1-q)e^{-\lambda(1-q)x}, \qquad (13)$$

where $\lambda = 1/\mathbb{E}(d)$ as before (i.e., the placement density of sensors). Substituting (8) and (13) into (7), it can be verified that K_L is given by (10) with λ replaced by $\lambda(1-q)$, and that the optimal placement density in the presence of sensor failures is

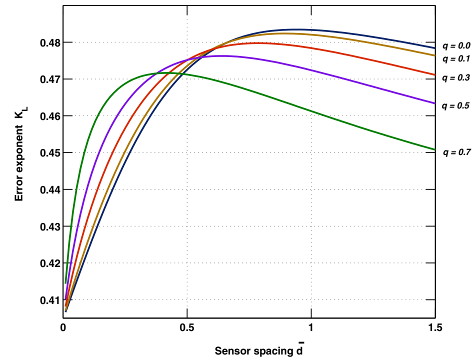
$$\lambda^* = \frac{1}{1-q}\sqrt{2A_1(A_0 + A_1)}. \qquad (14)$$

Thus, as q increases, sensors should be placed more densely.

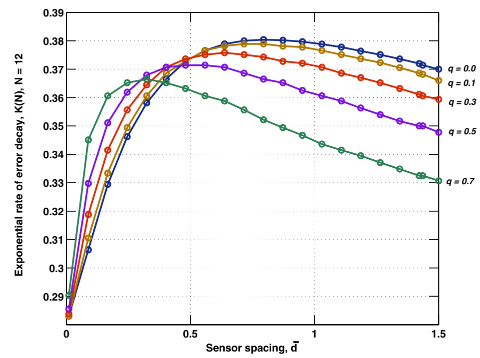
4. STATIONARY MARKOV MODEL

The stationary Markov model is given as described in the first paragraph of Section 3 by redefining the impulsive noise term as

$$z_{j,k} \overset{i.i.d.}{\sim} \mathcal{N}(0,\ \sigma_j^2(1 - a_{j,k}^2)), \qquad (15)$$



(a)



(b)

Fig. 3. Bayesian detection performance versus the sensor spacing $\bar{d}$ when $q \in \{0.0, 0.1, 0.3, 0.5, 0.7\}$ for (a) the error exponent K_L, and (b) the true error rate $K(N)$ with $N = 12$. Parameters: $\pi_0 = \pi_1 = \sigma_{0,I}^2 = \sigma_{1,I}^2 = 0.5$, $\sigma_0^2 = 1$, $\sigma_1^2 = 5$, $A_0 = 4$, and $A_1 = 0.1$.

and secondly, by taking the special case that $\sigma_{j,I}^2 = \sigma_j^2,\ j \in \{0, 1\}$. It easy to verify that VAR $\left(s_k^2\right) = \sigma_j^2$, for all k, under $\mathcal{H}_j$. Thus, this Gauss-Markov model is stationary. We emphasize that this model is *not* a special case of the non-stationary model of Section 3. For example, (15) introduces dependency of the random quantity d_k into $z_{j,k}$, which was not the case in Section 3.

4.1. Derivation of the Error Exponent

We start by evaluating the log likelihood ratio. Following a procedure similar to the non-stationary case, we get

$$\ln\frac{p_0(\mathbf{s}, \mathbf{d})}{p_1(\mathbf{s}, \mathbf{d})} = \frac{1}{2}\ln\frac{\sigma_1^2}{\sigma_0^2} + \frac{s_1^2}{2\sigma_1^2} - \frac{s_1^2}{2\sigma_0^2} + \frac{1}{2}\sum_{k=2}^{N}\ln\frac{\sigma_1^2(1-a_{1,k}^2)}{\sigma_0^2(1-a_{0,k}^2)} + \frac{1}{2}\sum_{k=2}^{N}\left\{\frac{(s_k - a_{1,k}s_{k-1})^2}{2\sigma_1^2(1-a_{1,k}^2)} - \frac{(s_k - a_{0,k}s_{k-1})^2}{2\sigma_0^2(1-a_{0,k}^2)}\right\}.$$

We take the expected value of the above under $\mathcal{H}_0$. We get

$$\mathbb{E}_{\mathcal{H}_0}\left(\ln\frac{p_0(\mathbf{s}, \mathbf{d})}{p_1(\mathbf{s}, \mathbf{d})}\right) = \frac{1}{2}\ln R + \frac{1}{2}\left(\frac{\sigma_0^2}{\sigma_1^2} - 1\right) + \frac{N-1}{2}\left\{\ln R - 1 + \mathbb{E}_d\left[\ln\frac{1-a_1^2}{1-a_0^2}\right]\right\} + \frac{\sigma_0^2}{2\sigma_1^2}\left\{(N-1)\mathbb{E}_d\left[\frac{1 + a_1^2 - 2a_0a_1}{1 - a_1^2}\right]\right\},$$

where we have omitted calculations for brevity. Taking the limit as $N \to \infty$, we find the error exponent to be

$$K_L = \frac{1}{2}\left\{ \ln R - 1 + \mathbb{E}_d\left[\ln\left(\frac{1-g_1(d)^2}{1-g_0(d)^2}\right)\right] + \frac{1}{R}\mathbb{E}_d\left[\frac{1-2g_0(d)g_1(d)+g_1(d)^2}{1-g_1(d)^2}\right]\right\}. \quad (16)$$

This expression is valid whether F_d is a continuous or discrete distribution. In the special case that d is deterministic, (16) matches the Neyman-Pearson error exponent that we previously derived in [10].

4.1.1. Example 2

We evaluate (16) for exponentially distributed sensors with independent sensor failures. The probability density of the spacing between two consecutive operational sensors is given by (13), where $\lambda = 1/\mathbb{E}(d)$ is the placement density of sensors. We use the correlation model (8). Substituting (8) and (13) into (16) and simplifying, we get

$$K_L = \frac{1}{2}\left\{ \ln R - \frac{R+1}{R} + \Psi\left(1+\frac{\lambda_q}{2A_0}\right) - \Psi\left(1+\frac{\lambda_q}{2A_1}\right) + \frac{\lambda_q}{RA_1}\left[\Psi\left(\frac{1}{2}+\frac{A_0+\lambda_q}{2A_1}\right) - \Psi\left(\frac{\lambda_q}{2A_1}\right)\right]\right\}, \quad (17)$$

where $\lambda_q \triangleq \lambda(1-q)$ and where $\Psi(x)$ is the Psi function [5, p.943].

When sensors are sparsely placed ($\lambda \ll 1$), consecutive signal samples approach statistical independence under both hypotheses. Therefore, we expect K_L to depend only on the signal powers, $\{\sigma_j^2\}$. Using the fact that $\Psi(x) = -\frac{1}{x} + O(x^0)$ [5, p.943], we can show that

$$\lim_{\lambda\to 0} K_L = \ln R + \frac{1}{R} - 1. \quad (18)$$

When $R = 1$, the error exponent is 0, as expected.

When sensors are densely placed ($\lambda \gg 1$), we use the asymptotic expansion, $\Psi(x) \to \ln(x)$ for large x [5, p.943], and find that

$$\lim_{\lambda\to\infty} K_L = \ln\left(R\frac{A_1}{A_0}\right) + \left(R\frac{A_1}{A_0}\right)^{-1} - 1. \quad (19)$$

The minimum w.r.t. R occurs when $R = A_0/A_1$. Note that $A_0/A_1 > 1 \quad (< 1)$ implies that the minimum occurs for $R > 1 \quad (< 1)$. This reflects the intuitive fact that detection is *harder* when the hypothesis with the more strongly correlated signal is also the hypothesis for which the signal variance is greater. In Figure 4, we plot K_L versus R for several "large" and "small" values of λ when $A_0/A_1 = 10$, $A_0 = 2$, and $q = 0$. Note that the limits as $\lambda \to 0$ and $\lambda \to \infty$ represent local minimizers of K_L in λ.

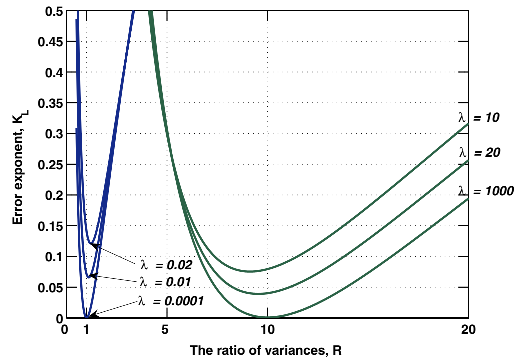


Fig. 4. The error exponent K_L versus R for several "large" ($\lambda \in \{10, 20, 1000\}$) and "small" ($\lambda \in \{0.0001, 0.01, 0.02\}$) values of λ when $A_0/A_1 = 10$, $A_0 = 2$, and $q = 0$.

5. DISCUSSION AND FUTURE WORK

We have studied error exponents for Bayesian detection of Gauss-Markov signals with random (i.e., ad-hoc) sensor spacing. For a summary of the paper, please refer to Section 1.3. We now discuss assumptions made in this work and detail avenues of further research.

The assumption that samples are collected along straight line can be relaxed. If sensors are not located in a straight line, one way to apply the results of this paper is as follows: Generalize $g_j(.)$ to specify the correlation as a function of the *Euclidean* distance separating two consecutively sampled sensors, and let F_d be the distribution on this Euclidean distance. The results (7) and (16) still hold with this new interpretation of F_d, and it would be interesting to see if there exist special cases for which these equations simplify to closed form expressions.We would like to investigate extensions of the model considered here to noisy sensor observations, and when $\{d_k\}$ is unknown at the fusion center.

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