Stability and Delay of Finite User Slotted ALOHA with Multipacket Reception

Vidyut Naware, Gökhan Mergen and Lang Tong[*]

School of Electrical Engineering
384 Frank H.T. Rhodes Hall
Cornell University
Ithaca, NY 14853
Tel:(607) 255-3500. Fax: (607) 255-9072.
Email: ltong@ece.cornell.edu

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[*] Corresponding Author

Abstract

The effect of Multipacket Reception (MPR) on stability and delay of slotted ALOHA based random access systems is considered. A general asymmetric MPR model is introduced and the MAC capacity region is specified. An explicit characterization of the ALOHA stability region for the two user system is given. It is shown that the stability region undergoes a *phase transition* from a concave region to a convex region bounded by lines as the MPR capability improves. It is also shown that after this phase transition, slotted ALOHA is optimal *i.e.,* the ALOHA stability region coincides with the MAC capacity region. Further, it is observed that there is no need for transmission control when ALOHA is optimal *i.e.,* ALOHA with transmission probability one is optimal. These results are extended to a symmetric $N > 2$ user ALOHA system, where it is shown that for a large class of symmetric MPR channels no transmission control is optimal from a stability viewpoint. This finding suggests that if the physical layer is even *reasonably* good, there is no need for sophisticated Medium Access Control protocols. Next, sufficient conditions for stability of slotted ALOHA for the asymmetric $N > 2$ case are provided. Finally, a complete characterization of average delay in capture channels for the two user system is given. In cases with non-zero capture probability, no transmission control is found to minimize delay for a subset of stable arrival rates. It is shown that in certain capture scenarios, no transmission control is delay-optimal for all stable arrival rates. Further, it is also shown that no transmission control is optimal for stability and delay *simultaneously* in the two user capture channel.

Index Terms

Wireless networks, Random access, Multipacket reception, Capacity, Stability, Delay, Slotted ALOHA, Scheduling.

I. Introduction

A. Motivation

It has been more than three decades since Abramson's landmark work on ALOHA [1]. Much of what we know about slotted ALOHA is based on the so-called collision model: a transmission is successful if and only if a single user transmits. While a deterministic collision model is accurate for wire-line communications, it is inadequate to model probabilistic receptions in wireless multiple access. Furthermore, advances in multiuser detection and space-time processing make it necessary to have a multipacket reception model that captures the ability of the receiver to decode simultaneous transmissions and the probabilistic nature of reception.

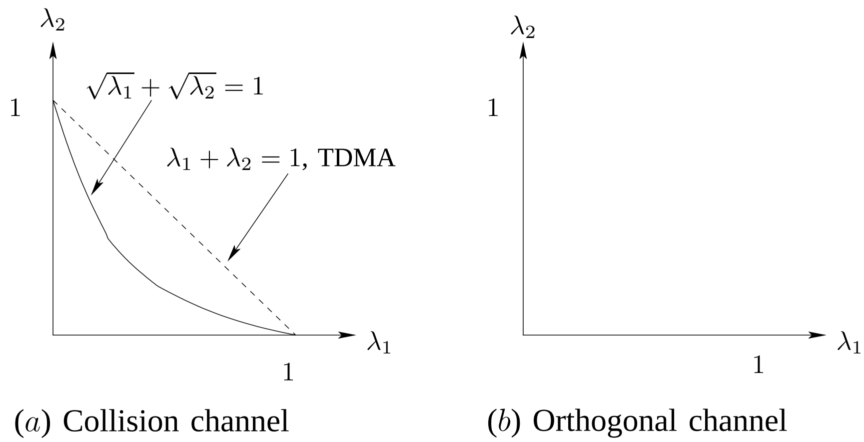


Fig. 1. Two user stability region of slotted ALOHA for the Collision Channel and Orthogonal Channels. Solid lines represent the boundary of the ALOHA stability region.

Insights into the effect of MPR on ALOHA can be gained by examining two extreme cases: the collision channel and the orthogonal channel. Figure 1 shows the ALOHA stability regions of the two-user system for these cases. By stability region we mean the set of arrival rates such that there exists retransmission probabilities that make the probability of buffer overflow arbitrarily small. For the collision model, the stability region is not convex; the increase of the maximum rate for one user implies an disproportionate decrease of the other. As a random access protocol, ALOHA is inferior to centralized TDMA since its stability region is contained inside that of TDMA. To stabilize any point in the rate region, transmission control is necessary by choosing transmission probabilities carefully. The onus of handling multiuser interference rests entirely with the random access protocol. The orthogonal channel, in contrast, models a physical layer that nullifies multiuser interference. As a result, the stability region takes the simple form of a unit square. There is no need for transmission control, and the rate for one user is independent of that of the other; ALOHA is optimal.

The orthogonal channel, of course, is not interesting for random access. What would be interesting are those cases when the multiuser interference affects the reception but not as severely as in the collision model. Can a distributed random access protocol such as ALOHA still be optimal? Is transmission control necessary? Is the stability region convex? A positive answer to the last question implies that proportional change of rates maintains stability. What can we say about the performance of ALOHA for the general N-user system?

B. Summary of Contributions

We consider a general multipacket reception model. For each scheduled transmission, this model specifies a probability measure on the event space. We first give a complete characterization of MAC capacity region. By MAC capacity we mean the maximum throughput achievable by any MAC protocol without considering queue stability. We show that this region is a convex hull of a set of marginal probabilities. In particular, the MAC capacity region is specified only by the marginal probabilities of success of individual users.

We consider next the ALOHA stability region. Obviously, the ALOHA stability region is contained in the MAC capacity region. As already shown in Figure 1, the ALOHA stability region is, in general, strictly smaller than the capacity region. We give a complete characterization for the two-user ALOHA system. We show that the stability region undergoes a distinct phase-transition, from a nonconvex region to a convex region bounded by lines, from a strict subset of the capacity region to the exact capacity region (thus ALOHA is optimal). Furthermore, there is no need for transmission control once ALOHA is optimal. The same results hold for the symmetrical N-user system which has indistinguishable users with equal arrival rates. An inner bound for the general asymmetrical N-user systems is provided.

For a given rate vector, there are usually many transmission probabilities that stabilize the system. It is thus interesting to find the transmission probability that minimizes the average delay. We provide a complete delay characterization for the capture model in a symmetrical two-user system. Any nonzero probability of capture leads to a set of rates for which no transmission control minimizes the delay. As the probability of capture increases, the region of rates for which no transmission control minimizes the delay increases. As soon as the stability region becomes convex, no transmission control is delay-optimal for all stable arrival rates.

C. Related Work

In spite of being such a simple random access protocol, queueing theoretic analysis of ALOHA turns out to be extremely difficult under the collision model. Tsybakov and Mikhailov [3] initiated the stability analysis of finite-user slotted ALOHA. They found sufficient conditions for stability of the queues in the system using the principle of stochastic dominance. They found the stability region for the two-user case explicitly. For the symmetric case (*viz*. equal arrival rates for all terminals), they gave the maximum stable throughput. Rao and Ephremides [4] explicitly used the principle of stochastic dominance to find inner bounds to the stability region for the $N > 2$ case. Szpankowski [5] found necessary and sufficient conditions for the stability of queues for a fixed transmission probability vector for the $N > 2$ case. However, this necessary and sufficient condition cannot be computed explicitly since it involves the stationary joint queue statistics, which cannot be computed. Later, Luo and Ephremides [6] introduced the concept of instability ranks in queues to obtain tight inner and outer bounds on the stability region for the $N > 2$ case. Interestingly, Anantharam [7] found the exact stability region of ALOHA for the finite user case, albeit with a specific correlated arrival process. All the above stability results were derived for the collision channel only. And to date there is no closed form characterization of the stability region for the $N > 2$ case (even for the collision channel with i.i.d. arrivals). The primary difficulty in analyzing this problem is the complex interactions amongst the queues.

The first attempt at analyzing ALOHA under multipacket reception was made by Ghez, Verdú and Schwartz in [8], [9] under the infinite-user single-buffer model. Their MPR model was symmetrical in which users were indistinguishable. A special case of the symmetrical MPR model, but for finite users, was analyzed by Sant and Sharma [10]. They found a sufficient condition for stability with no transmission control. Adireddy and Tong [11] considered the effect of having knowledge of fading at the transmitters on the design of ALOHA. They showed that significant gains can be made by allowing the transmission probability to be a function of the channel state (as opposed to conventional power control). However, the MPR model that they used was symmetric with respect to the users while ours is not. A study of stability and capacity of general wireless networks for MPR models was presented in [12] where the MAC stability and capacity regions were characterized. Protocols that exploit MPR have been proposed [13], [14], [11].

At this point we would like to mention that a different line of analysis is based on the infinite-user single-buffer model [8], [9]. Such a model does not accommodate asymmetric arrivals and is generally considered to give a performance lower bound.

The remainder of this paper is organized as follows. In Section II, we specify the system model. In Section III, we define the notion of capacity region and in Section IV we define stability region. In Section V, we derive the stability region for the two user case. We also characterize some interesting properties of this region. In Section VI, we provide stability results for the symmetric MPR case with $N > 2$. We also give sufficient conditions for stability for the asymmetric MPR case with $N > 2$. In Section VII, we apply our analytical results to three different receiver structures *viz.*, Decorrelating, Matched filter and MMSE and compare their performance in terms of the stability region to gain some insights. In Section VIII, we find expressions for delay and the optimizing transmission probability for the $N = 2$ case for a subclass of MPR reception models. Finally, we conclude in Section IX.

II. System Model

The system consists of N users communicating with a common receiver. Each user has an infinite buffer for storing arriving and backlogged packets. The channel is slotted in time and a slot duration equals the packet transmission time. Packets are assumed to be of equal length for all the users. The arrivals at the ith queue ($i \in \{1, 2, \ldots, N\} \triangleq \mathcal{M}$) are independent and identically distributed from slot to slot with mean λ_i. Arrival processes are assumed to be independent from user to user. If the ith users' buffer is nonempty, he transmits a packet with probability p_i in a slot.

A multiuser physical layer is assumed that allows the receiver to receive multiple packets simultaneously. Specifically, suppose that the set $\mathcal{S} \subseteq \mathcal{M}$ of users transmit in a slot, then we define for $\mathcal{R} \subseteq \mathcal{S}$, the conditional probability of reception by

$$q_{\mathcal{R},\mathcal{S}} = \Pr\{\textit{only} \text{ packets from } \mathcal{R} \text{ are successfully received} \mid \mathcal{S} \text{ transmits}\}. \tag{1}$$

We assume that packet receptions are independent from slot to slot. Note that our reception model completely defines the probability space of packet receptions. The MPR reception model defined in [8] is symmetric with respect to the users and a special case of the above model. It follows that the *marginal* probability of success of $\mathcal{R}$ given that set $\mathcal{S}$ of users transmit is given by

$$q_{\mathcal{R}|\mathcal{S}} = \sum_{\mathcal{U}:\mathcal{R}\subseteq\mathcal{U}\subseteq\mathcal{S}} q_{\mathcal{U},\mathcal{S}}. \tag{2}$$

For example, consider the two user case $N = 2$. Then for $i = 1, 2$,

$$\begin{aligned} q_{i,\{i\}} &= \Pr\{\text{user } i \text{ is successful} \mid \text{only user } i \text{ transmits}\}, \\ q_{i,\{1,2\}} &= \Pr\{\textit{only} \text{ user } i \text{ is successful} \mid \text{both users transmit}\}, \\ q_{\{1,2\},\{1,2\}} &= \Pr\{\text{both users are successful} \mid \text{both users transmit}\} \end{aligned} \tag{3}$$

and the marginal probabilities of success are ($i = 1, 2$)

$$q_{i|\{i\}} = q_{i,\{i\}}, \quad q_{i|\{1,2\}} = q_{i,\{1,2\}} + q_{\{1,2\},\{1,2\}}. \tag{4}$$

We assume that the receiver gives an instantaneous feedback of all the packets that were successful in a slot at the end of the slot to all the users. The users remove successful packets from their buffers while unsuccessful packets are retained. It should be clear that the probabilities $q_{\mathcal{R},\mathcal{S}}$ are a function of the receiver front-end which will be employed by the receiver to “separate” users’ signals.

Let Q_j^t represent the queue length at the jth buffer at the beginning of time slot t. Under the above system model, the N-dimensional process $\mathbf{Q}^t = (Q_1^t, Q_2^t, \cdots, Q_N^t)$ is a Markov Chain. The transition probability matrix of the Markov chain $\mathbf{Q}^t$ can be computed using the reception probabilities given by (1). Under mild conditions (for instance, $q_{i|i} > 0$ for all $i \in \mathcal{M}$) $\mathbf{Q}^t$ is irreducible and aperiodic. We will assume $\mathbf{Q}^t$ to be an irreducible and aperiodic Markov Chain throughout this paper. The queue evolution for the jth queue has a well known form [5]

$$Q_j^{t+1} = [Q_j^t - Y_j^t]^+ + \beta_j^t, \tag{5}$$

where β_j^t is the number of arrivals during the tth slot to the jth user with $\mathbb{E}(\beta_j^t) = \lambda_j < \infty$ and $[x]^+$ denotes $\max\{0, x\}$. The Bernoulli random variable Y_j^t denotes departures[1] from queue j in time slot t. For the ALOHA model we are considering, Y_j^t can be represented as

$$Y_j^t = R_j^t \sum_{\mathcal{S} \subseteq \mathcal{M} \setminus \{j\}} \mathbf{1}[\,\mathcal{S} = \{k : R_k^t \mathbf{1}[Q_k^t > 0] = 1\}]\, \mathbf{1}[Z_j^t(S) = 1], \tag{6}$$

where $R_j^t \overset{i.i.d}{\sim}$ Bernoulli(p_j) for $1 \leq j \leq N$ denote the “coin tossing” outcomes that determine transmission attempts, $\mathbf{1}[\cdot]$ the indicator function, and $Z_j^t(\mathcal{S})$ a Bernoulli random variable that

[1]The process $\{Y_j^t\}_{t=1}^{\infty}$ represents departures in the sense that $Y_j^t = 1$ implies that a packet from queue j was successfully received in slot t *only* when $Q_j^t > 0$. However, we could have $Y_j^t = 1$ even when $Q_j^t = 0$. Nonetheless, equation (5) still models the queue evolution process correctly [4].

takes the value 1 in slot t with probability $q_{j|\{j\}\cup\mathcal{S}}$. Thus, the event $\{Z_j^t(\mathcal{S}) = 1\}$ captures all possible successful MPR events which include queue j.

III. MAC CAPACITY

For the reception model defined by (1), we now define the notion of *capacity region* ($\mathscr{C}$) of the network. Suppose that at $t = 0$ all users in the network have infinitely many packets to send to the receiver. One may ask what possible long term rates the reception model specified by (1) can *support* or achieve with optimal centralized scheduling. Here, we neglect the effects of source burstiness and thus the long term "achievable" rates depend only on the reception model.

Let $\mathcal{R}_S(t)$ be the set of successful transmissions when the set of users $\mathcal{S}_S(t)$ transmit in slot t under scheduling policy S. We allow the scheduling policy to be a function of the history of the network *viz.*, all the past arrivals and the packet success outcomes. The scheduling policy can be randomized as well.

Definition 1: A rate $\boldsymbol{\lambda} = (\lambda_1, \cdots, \lambda_N)$ is called *achievable* if there exists a scheduling policy (S^*) with delivery rate greater than $\boldsymbol{\lambda}$, *i.e.*,

$$\liminf_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} \mathbf{1}(i \in \mathcal{R}_{S^*}(t)) \geq \lambda_i \qquad \forall i \quad a.s. \tag{7}$$

Capacity region ($\mathscr{C}$) is the closure of the set of all achievable rates.

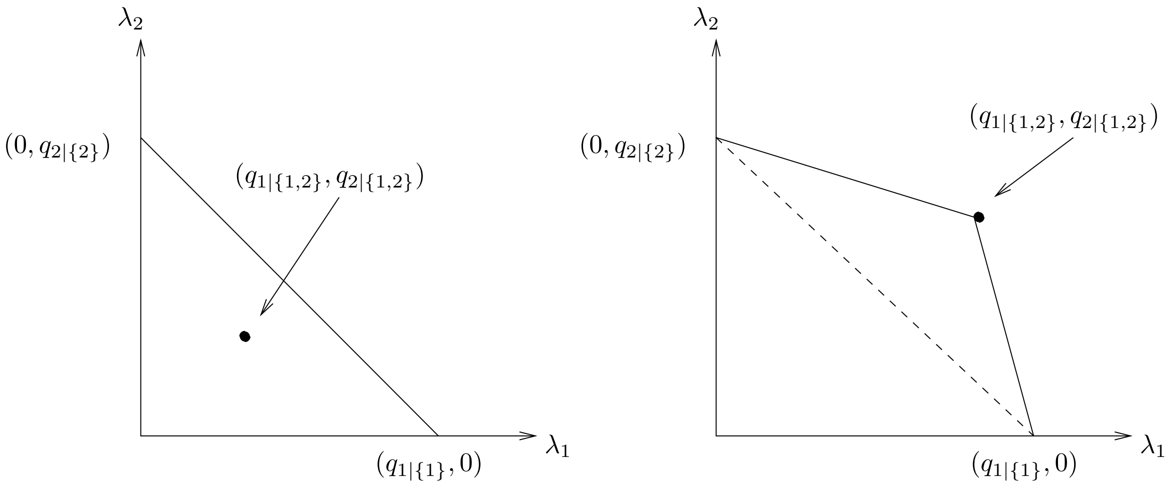
This notion of "achievable" rate has been used before in [15], [12]. The next theorem provides a simple way to compute $\mathscr{C}$ in terms of the marginal probabilities of success of each user.

Theorem 1: A rate $\boldsymbol{\lambda}$ is achievable if and only if there exists a probability measure $\{p(\mathcal{S}) : \mathcal{S} \subseteq \mathcal{M}\}$ such that $\forall i \in \mathcal{M}$,

$$\lambda_i \leq \sum_{\mathcal{S}\subseteq\mathcal{M}} q_{i|\mathcal{S}}\, p(\mathcal{S}). \tag{8}$$

Proof: Refer to the Appendix.

The above result shows that $\mathscr{C}$ is the convex hull of the N tuples consisting of the marginal reception probabilities of the users in all possible 2^N transmission scenarios. Intuitively, the achievability part of the proof follows by observing that if a scheduler chooses the subset of transmitting users $\mathcal{S}$ with probability $p(\mathcal{S})$ i.i.d. in every slot, then $\boldsymbol{\lambda}$ satisfying (8) is achievable. Note that a direct consequence of the above theorem is that the capacity region is convex.



(a) $\frac{q_{1|\{1,2\}}}{q_{1|\{1\}}} + \frac{q_{2|\{1,2\}}}{q_{2|\{2\}}} < 1$ (b) $\frac{q_{1|\{1,2\}}}{q_{1|\{1\}}} + \frac{q_{2|\{1,2\}}}{q_{2|\{2\}}} > 1$

Fig. 2. $\mathscr{C}$ (two user case) under different reception models

Figure 2 shows the two user capacity region for two different reception models. Clearly, the convex hull can take only two possible forms; either it is a triangle (Case (a)) or it is a quadrilateral (Case (b)). For Case (a), optimal scheduling is equivalent to TDMA where to achieve any rate in the capacity region, it suffices to allow only one user to transmit in a slot. On the other hand in Case (b), the scheduler has to consider allowing both users to transmit simultaneously to achieve some rate pairs.

IV. MAC Stability Region

Before we proceed to derive some of the results of the next section, a few definitions are in order. We use the definition of stability used by Szpankowski [5].

Definition 2: A multidimensional stochastic process, $\mathbf{Q}^t = (Q_1^t, \ldots, Q_N^t)$ is *stable* if for $\mathbf{x} \in \mathbb{N}^N$ the following holds

$$\lim_{t\to\infty} \Pr\{\mathbf{Q}^t < \mathbf{x}\} = F(\mathbf{x}) \text{ and } \lim_{\mathbf{x}\to\infty} F(\mathbf{x}) = 1. \tag{9}$$

If a weaker condition holds *viz.*,

$$\lim_{\mathbf{x}\to\infty} \liminf_{t\to\infty} \Pr\{\mathbf{Q}^t < \mathbf{x}\} = 1, \tag{10}$$

then the process is called *substable*. Further, the process is said to be unstable if it is not substable.

The related concepts of stability and substability have been well studied (see [5], [6]). For a queueing system, stability can be interpreted as the convergence of the queue lengths in distribution to a proper random variable (*viz.,* a random variable that is finite with probability one) or, equivalently, the existence of a proper limiting distribution. As mentioned before, with ALOHA, the queue process $\mathbf{Q}^t$ is an aperiodic and irreducible Markov Chain on a countable state space. It can be shown that for $\mathbf{Q}^t$, the notions of stability and substability are equivalent and stability is equivalent to the existence of a unique stationary distribution (see [16]). Though the transition matrix of $\mathbf{Q}^t$ depends on the reception probabilities given by (1), we will see that the stability properties of $\mathbf{Q}^t$ can be characterized with only the marginal probabilities of success of users ($\{q_{i|\mathcal{T}} : \mathcal{T} \subseteq \mathcal{M},\ i = 1, \cdots, N\}$).

It would be natural to expect the stability of a queueing system to depend on the average arrival rate and average service rate. This intuition is made concrete by the Loynes' theorem [17] which says that if the arrival and departure process of a queueing system are strictly stationary and ergodic then (i) the queueing system is stable if the average arrival rate is less than the average departure rate and (ii) the queueing system is unstable if the average arrival rate exceeds the average departure rate.

It follows from Loynes' theorem that the arrival rates $\boldsymbol{\lambda} = [\lambda_1, \lambda_2, \ldots, \lambda_N]$ (*i.e.,* the first order arrival statistics) and the packet reception model would completely characterize the stability of queues. This motivates the following characterization of stability.

Definition 3: For an N user multiple access system with a given MAC protocol, the stability region is defined as the closure[2]of the set of arrival rates $\boldsymbol{\lambda} = [\lambda_1, \lambda_2, \ldots, \lambda_N]$ for which the queues in the system are stable.

In particular, for an N user slotted ALOHA system, the stability region is defined as the set of arrival rates $\boldsymbol{\lambda} = [\lambda_1, \lambda_2, \ldots, \lambda_N]$ for which there exists a transmission probability vector $\mathbf{p} = [p_1, p_2, \ldots, p_N]$ such that the queues in the system are stable. We will denote the stability region of ALOHA by $\mathscr{S}_{\text{ALOHA}}$. We define the stability region ($\mathscr{S}$) to be the union of the stability regions over all MAC protocols (for the reception model given by (1)).

[2]Generally, it is difficult to characterize stability on the boundary of the stability region. The set operation of closure allows us to conveniently get around stability properties of points on the boundary of the stability region

The capacity region characterizes the set of *departure* rates that are supported by centralized scheduling whereas the stability region provides the set of stable *arrival* rates with all MAC protocols. Here, note that we consider MAC protocols with memory *viz.*, the MAC scheme can allow users to transmit based on the past history of outcomes. Intuitively, we expect the stability region of any MAC protocol to be contained within the capacity region since in a stable system, the arrival rate is equal to the departure rate[3].

Theorem 2: For the N user random access system with reception model specified by (1), $\mathscr{S} \subseteq \mathscr{C}$.

Proof: Refer to the Appendix.

Thus, $\mathscr{C}$ provides a simple easily computable upper bound to $\mathscr{S}_{\text{ALOHA}}$. However, unlike the capacity region, the stability region of ALOHA is not easy to characterize. We have the following relation: $\mathscr{S}_{\text{ALOHA}} \subseteq \mathscr{S} \subseteq \mathscr{C}$.

V. Stability and Optimality of ALOHA for $N = 2$

A. Stability region of ALOHA

We first find the stability region for the $N = 2$ case for the general reception model given by (3). We will show that only the marginal probabilities given by (4) are sufficient to characterize the stability region of ALOHA.

Define $Q_1 \triangleq q_{1|\{1\}} - q_{1|\{1,2\}}$, $Q_2 \triangleq q_{2|\{2\}} - q_{2|\{1,2\}}$ and $\mathbf{q} \triangleq [q_{1|\{1\}}, q_{2|\{2\}}, q_{1|\{1,2\}}, q_{2|\{1,2\}}]$. Thus, Q_1 and Q_2 denote the difference between the (conditional) probability of success in the absence of interference and the (conditional) probability of success in the presence of interference for the users. For the collision channel, $Q_1 = Q_2 = 1$ whereas for orthogonal channels, $Q_1 = Q_2 = 0$.

To find the stability region, we first find the stability region of the ALOHA system for a *fixed* transmission probability vector $\mathbf{p}(= [p_1, p_2])$. The following lemma gives us exactly that.

Lemma 1: If $Q_1 \geq 0$ and $Q_2 \geq 0$, the stability region of slotted ALOHA for the general packet reception model for a given $[p_1, p_2]$ ($\mathscr{S}_{\text{ALOHA}}(\mathbf{p})$) is given by

$$\lambda_1 \leq p_1 q_{1|\{1\}} - \frac{p_1 p_2 \lambda_2 Q_1}{\lambda_2^*}, \text{ for } \lambda_2 \leq \lambda_2^* \tag{11}$$

[3]In fact in [12], it is shown that under certain conditions on the reception model, $\mathscr{S} = \mathscr{C}$.

and

$$\lambda_2 \leq p_2 q_{2|\{2\}} - \frac{p_1 p_2 \lambda_1 Q_2}{\lambda_1^*}, \text{ for } \lambda_1 \leq \lambda_1^*, \tag{12}$$

where,

$$\lambda_1^* = p_1 q_{1|\{1\}} - p_1 p_2 Q_1 \text{ and } \lambda_2^* = p_2 q_{2|\{2\}} - p_1 p_2 Q_2.$$

Proof: We use the idea of stochastic dominance and an argument similar to that by Rao and Ephremides [4]. Refer to the Appendix for details. □

The stability region of ALOHA with fixed $[p_1, p_2]$ for the following well known cases are:

(a) Orthogonal channel: In this case $Q_1 = Q_2 = 0$. By Lemma 1, the stability region is the two dimensional box in the nonnegative quadrant bounded by the lines $\lambda_1 = 0$, $\lambda_2 = 0$, $\lambda_1 = p_1 q_{1|\{1\}}$ and $\lambda_2 = p_2 q_{2|\{2\}}$.

(b) Collision channel: Since $Q_1 = Q_2 = 1$ in this case, $\lambda_1^* = p_1(1-p_2)$ and $\lambda_2^* = p_2(1-p_1)$. The stability region is

$$\lambda_1 \leq p_1 \left(1 - \frac{\lambda_2}{(1-p_1)}\right), \text{ for } \lambda_2 \leq p_2(1-p_1) \tag{13}$$

and

$$\lambda_2 \leq p_2 \left(1 - \frac{\lambda_1}{(1-p_2)}\right), \text{ for } \lambda_1 \leq p_1(1-p_2). \tag{14}$$

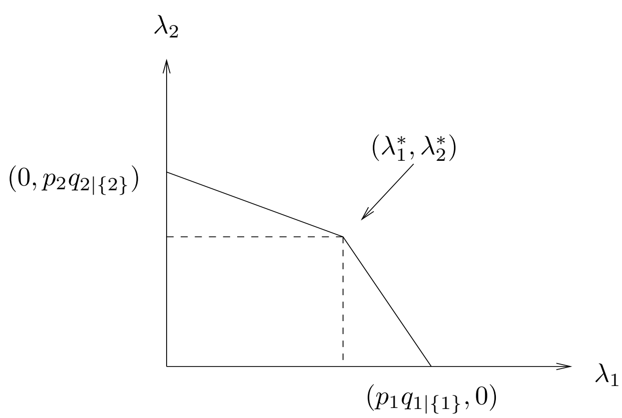


Fig. 3. Stability region for a fixed transmission probability vector $[p_1, p_2]$

Irrespective of the reception model, the stability region for a given $\mathbf{p}$ has a form as shown in Figure 3. The conditions $Q_1 \geq 0$ and $Q_2 \geq 0$ are equivalent to the probability of success of any user in the presence of interference (from the other user) be no greater than the probability of success in the absence of interference—a reasonable and practical assumption. Note that although Figure 3 shows the stability region for a fixed $\mathbf{p}$ to be convex but it need not be convex as $\mathbf{p}$ varies over $[0,1]^2$. It follows from Lemma 1 directly that the stability region of ALOHA in the two user case depends only on the marginal probabilities of success of the users since Q_1 and Q_2 depend only on the marginals.

Using

$$\mathscr{S}_{\text{ALOHA}} = \bigcup_{\mathbf{p}\in[0,1]^2} \mathscr{S}_{\text{ALOHA}}(\mathbf{p}),$$

we give a complete description of the stability region of ALOHA with this Lemma.

Lemma 2: If $Q_1 \geq 0$ and $Q_2 \geq 0$, then the stability region of slotted ALOHA ($\mathscr{S}_{\text{ALOHA}}$) for the general reception model is given by $\mathcal{R}_1 \cap \mathcal{R}_2$ where

$$\begin{aligned}\mathcal{R}_1 \triangleq \{(\lambda_1,\lambda_2) : (\lambda_1,\lambda_2) \geq (0,0),\ (\lambda_1,\lambda_2) \text{ lies}\\ \text{below the curve } \lambda_2 = f(\lambda_1; q_{1|\{1\}}, q_{2|\{2\}}, Q_1, Q_2)\}\end{aligned} \tag{15}$$

and

$$\begin{aligned}\mathcal{R}_2 \triangleq \{(\lambda_1,\lambda_2) : (\lambda_1,\lambda_2) \geq (0,0),\ (\lambda_1,\lambda_2) \text{ lies}\\ \text{below the curve } \lambda_1 = f(\lambda_2; q_{2|\{2\}}, q_{1|\{1\}}, Q_2, Q_1)\},\end{aligned} \tag{16}$$

where,

$$f(\lambda;\alpha,\beta,\gamma,\delta) = \begin{cases} \beta - \frac{\lambda\delta}{\alpha-\gamma}, & \lambda \in \mathcal{I}_1 \\ \frac{(\sqrt{\alpha\beta}-\sqrt{\lambda\delta})^2}{\gamma}, & \lambda \in \mathcal{I}_2, \end{cases} \tag{17}$$

where,

$$\mathcal{I}_1 = [0, \frac{\beta(\alpha-\gamma)^2}{\alpha\delta}] \text{ and } \mathcal{I}_2 = (\frac{\beta(\alpha-\gamma)^2}{\alpha\delta}, \frac{\alpha\beta}{\delta}]. \tag{18}$$

If either Q_1 or Q_2 equals zero, then we assume $\frac{1}{0} = \infty$ and the result still holds.

Proof: Refer to the Appendix.

We note a few interesting things about the stability region. First, the function f characterizing the stability region in (17) is linear for some part of the domain $[0, \lambda_{1c})$ and is strictly convex in the remainder of its domain as illustrated in Figure 4. The stability region for the two user collision channel can be found as a special case with $q_{1|\{1\}} = 1$, $q_{2|\{2\}} = 1$, $q_{1|\{1,2\}} = 0$ and $q_{2|\{1,2\}} = 0$ and it is bounded by the curve $\sqrt{\lambda_1} + \sqrt{\lambda_2} = 1$, which is strictly convex everywhere. In fact, it is easy to see from Figure 4 that the interval where f is linear has non-zero Lebesgue

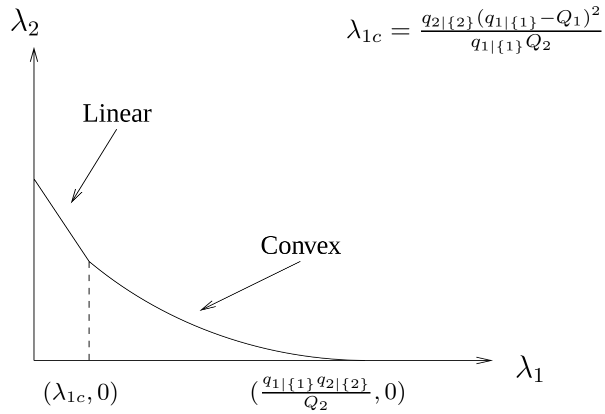


Fig. 4. The appearance of $f(\lambda_1)$ as a function of λ_1.

measure as soon as there is a nonzero probability of success in the presence of interference *i.e.* $q_{1|\{1\}} - Q_1 = q_{1|\{1,2\}} > 0$. Thus, there is a characteristic *change* in the structure of the stability region as soon as we have multipacket reception. Second, we see that there is a symmetry in the way the two regions $\mathcal{R}_1$ and $\mathcal{R}_2$ are defined in terms of the function f.

We now provide one of the main results that gives a structural characterization of the ALOHA stability region.

Theorem 3: Let $Q_1 \geq 0$ and $Q_2 \geq 0$. Assume $\mathbf{q} > 0$ *i.e.*, non-zero probability of success in the presence and absence of interference. Then, the following are equivalent.

1) $\mathscr{S}_{\text{ALOHA}}$ is convex.
2) $\mathscr{S}_{\text{ALOHA}}$ is bounded by lines.
3) The marginal reception probabilities satisfy

$$\mathcal{D}(\mathbf{q}) \triangleq \frac{q_{1|\{1,2\}}}{q_{1|\{1\}}} + \frac{q_{2|\{1,2\}}}{q_{2|\{2\}}} \geq 1. \tag{19}$$

4) $\mathscr{S}_{\text{ALOHA}} = \mathscr{S} = \mathscr{C}$.
5) $\mathbf{p} = [1, 1]$ is optimal in the sense that $\mathscr{S}_{\text{ALOHA}} = \mathscr{S}_{\text{ALOHA}}([1, 1])$.

If $\mathscr{S}_{\text{ALOHA}}$ is non-convex, then it is bounded by lines close to the axes and by a strictly convex function in the interior.

Proof: Refer to the Appendix.

Figure 5 shows the stability regions characterized by the $\mathbf{q}$ vector as given by Theorem 3. For the collision channel, the stability region is non-convex and bounded by a strictly convex curve.

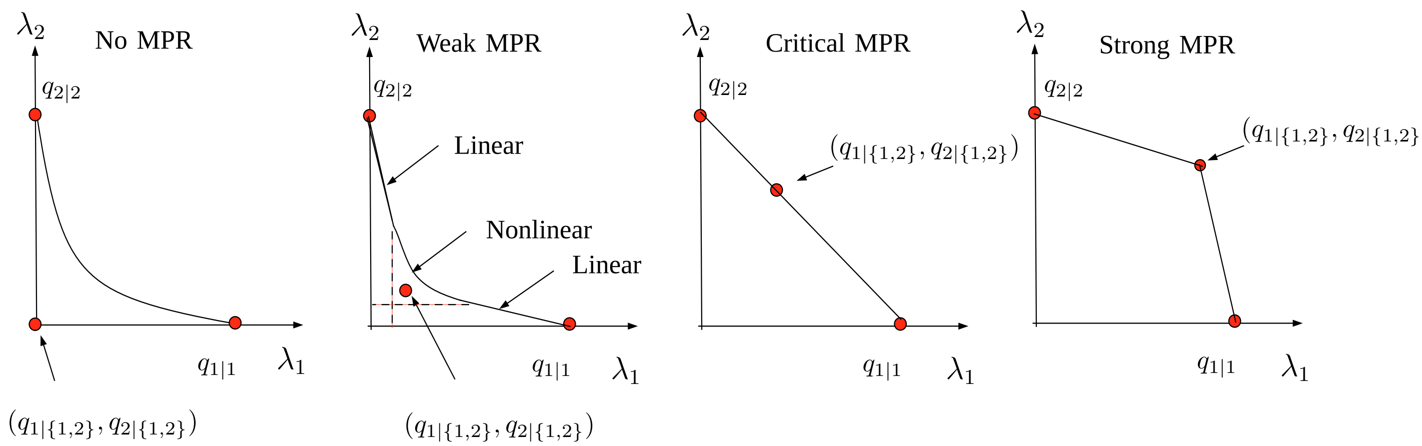


Fig. 5. $\mathscr{S}_{\text{ALOHA}}$ for different reception models with $q_{1|\{1\}}$ and $q_{2|\{2\}}$ fixed.

As soon as there is (weak) MPR, the stability region is bounded by lines near the axes and a non-linear strictly convex function elsewhere. After a certain critical MPR level ($\mathcal{D}(\mathbf{q}) = 1$) is reached, the stability region becomes convex and is bounded by lines. Thus, there is a critical point for the $\mathbf{q}$ vector at which the behavior of the stability region makes a phase transition from a very complex form to a much more simpler form (bounded by lines). Further, this critical point depends only on the sum of the ratios of probability of success of users in the presence of interference to that in the absence of interference.

The condition of the stability region being bounded by lines and being convex corresponds to a regime in which when one user increases his rate, the other user's maximum supportable rate decreases linearly, and that too at a rate which is low till a certain point and then suddenly increases. Another interpretation is that when the stability region is convex then higher sum rates can be achieved. In addition, when the stability region is convex we know that if two rate pairs are stable then any rate pair lying on the line segment joining those two rate pairs is also stable. When equality holds in equation (19), the stability region is a triangle as shown in Figure 5. All the rate pairs in this region can be stabilized by TDMA schemes (even in a collision channel). Thus, the condition $\frac{q_{1|\{1,2\}}}{q_{1|\{1\}}} + \frac{q_{2|\{1,2\}}}{q_{2|\{2\}}} > 1$ gives us the regime in which a distributed strategy like slotted ALOHA can do better than a TDMA scheme.

When $\mathscr{S}_{\text{ALOHA}}$ is not bounded by lines, it has a much more complex form. This is also the

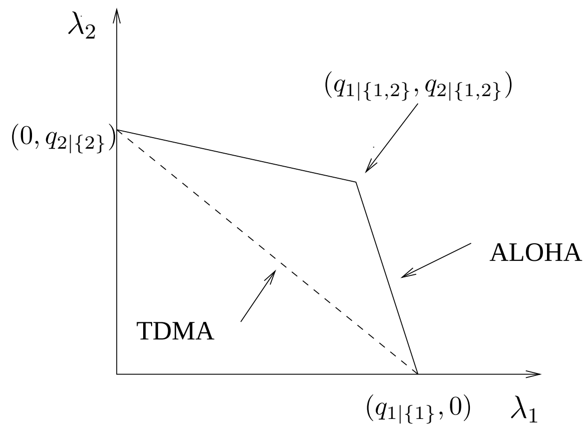


Fig. 6. $\mathscr{S}_{\text{ALOHA}}$ when it is convex.

regime in which the stability region is not convex. In this regime, when one user increases his rate the other user's rate has to be reduced drastically in order to keep the system stable.

B. Optimality of ALOHA and Dumb Scheduling

Equivalence conditions three and four in Theorem 3 specify the regime of MPR capability where slotted ALOHA is optimal. $\mathscr{S}_{\text{ALOHA}} = \mathscr{S}$ implies that ALOHA can stabilize all rates that can be stabilized by *any* centralized or decentralized MAC protocol. Note that the point from where slotted ALOHA is optimal coincides with the phase transition point of the ALOHA stability region.

In order to stabilize a rate within the stability region of ALOHA, one has to choose an appropriate transmission probability $\mathbf{p}$ which, in general is a function of the arrival rate. But the surprising observation when the stability region is convex, is that $\mathscr{S}_{\text{ALOHA}} = \mathscr{S}_{\text{ALOHA}}([1,1])$. This implies that when the stability region is convex, both users should always transmit packets (if they have any) to stabilize *any* stabilizable rate and no transmission control is required. We call this degenerate instance of ALOHA "Dumb Scheduling". Note that with centralized scheduling, to stabilize a particular rate the scheduler has to allocate a proportion of time for each possible subset of transmitting users. But the above result implies that there is no need for "scheduling" any transmissions. The strategy—transmit if you have packets—will do. The reason for this is that the users' queues empty out ever so often as a result of which there is a proportion of time when the users are transmitting alone. This *pseudo* scheduling of users automatically takes care of stabilizing the queues for the particular arrival rates.

The implication for cross layer design is clear—if we can design a *reasonably* strong physical layer, then there is no need for a sophisticated MAC layer. Intuitively it is quite clear that as the ability of the physical layer to orthogonalize users increases, then the need for random access protocols doesn't arise. But, surprisingly we find that the point at which we could dispense the MAC layer comes well before we have an ideal physical layer. Equation (19) gives both the metric for measuring the MPR capability and the condition under which the MAC layer is dispensable.

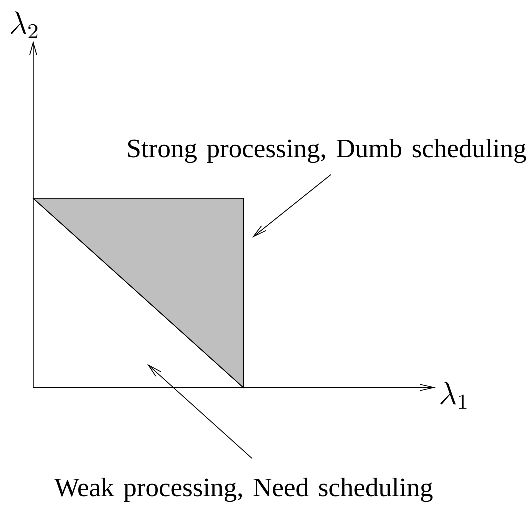


Fig. 7. Regions showing optimal allocation of resources to PHY and MAC layer for all MPR models

Figure 7 shows how the knowledge of MPR capability can help in designing a better MAC layer. In a low MPR regime, the PHY layer is weak; a larger amount of resources should be allocated to the MAC layer. On the other hand, if we allocate more resources to the PHY layer (with advanced signal processing) thereby guaranteeing a strong MPR channel, no resources are needed at the MAC layer; dumb scheduling is optimal.

VI. STABILITY OF ALOHA FOR THE $N > 2$ CASE

Little progress has been made in giving an exact characterization of the stability region of ALOHA for the $N > 2$ case. In this section, for a symmetric MPR channel, we provide conditions under which Dumb scheduling is optimal amongst all MAC protocols. For the more general asymmetric MPR channel, we provide sufficient conditions for stability.

A. Symmetric MPR, Symmetric arrivals case

For completeness, we first provide the symmetric MPR reception model introduced by Ghez, Verdu and Schwartz [8]. Multipacket reception is parametrized by a matrix **C** whose entries are given by

$$C_{n,k} = \text{ Pr}\{k \text{ packets are successfully received } | n \text{ packets are transmitted}\}. \tag{20}$$

Thus, we can write **C** as

$$\mathbf{C} = \begin{pmatrix} C_{1,0} & C_{1,1} & & \\ C_{2,0} & C_{2,1} & C_{2,2} & \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{21}$$

This channel model is general enough to model the collision channel and the capture channel as special cases. The corresponding MPR matrices for the collision channel, and the capture channel are, respectively,

$$\begin{pmatrix} 0 & 1 & 0 & \dots \\ 1 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & \dots \\ 1-b_2 & b_2 & 0 & \dots \\ 1-b_3 & b_3 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where b_i denotes the probability of capture given i simultaneous transmissions. We also define

$$C_n \triangleq \sum_{k=1}^{n} kC_{n,k}$$

which is the expected number of correctly received packets given n packets are transmitted. By symmetry of this reception model, it follows that C_i/i is the marginal probability of success of any one of the i users that transmit in a slot.

Consider an N user symmetric system with each user having an arrival rate $\frac{\lambda}{N}$. Let $\mathbf{Q}^t$ be the N tuple representing the queue lengths of users at time t. Given a reception model and a MAC protocol, we can define the maximum stable throughput to be the supremum of all arrival rates λ such that $\mathbf{Q}^t$ is stable. For example, the maximum stable throughput of ALOHA for a given transmission probability p (denoted by $\rho_{\text{ALOHA}}(p)$) is the supremum of all arrival rates λ such that $\mathbf{Q}^t$ is stable. Further, we define the maximum stable throughput of ALOHA to be

$$\rho_{\text{ALOHA}} = \sup_{p\in[0,1]} \rho_{\text{ALOHA}}(p).$$

Let ρ denote the supremum of the maximum stable throughput over all MAC protocols. By definition, $\rho_{\text{ALOHA}} \leq \rho$. We also have

Corollary 1 (to Theorem 2): For the N user symmetric system with symmetric MPR reception model given by (20),

$$\rho \leq \max\{C_1, C_2, \cdots, C_N\}. \tag{22}$$

Proof: By Theorem 2 for any stable arrival rates $\{\lambda_i = \frac{\lambda}{N} : \ i \in \mathcal{M}\}$, there exists a probability measure $\{p(\mathcal{S}) : \ \mathcal{S} \subseteq \mathcal{M}\}$ such that,

$$\lambda_i = \frac{\lambda}{N} \leq \sum_{\mathcal{S}} q_{i|\mathcal{S}} p(\mathcal{S}) \mathbf{1}[i \in \mathcal{S}].$$

Summing over i gives

$$\begin{aligned}
\lambda &\leq \sum_{i=1}^{N} \sum_{\mathcal{S}} q_{i|\mathcal{S}} p(\mathcal{S}) \mathbf{1}[i \in \mathcal{S}] \\
&= \sum_{i=1}^{N} \sum_{\mathcal{S}} \frac{C_{|\mathcal{S}|}}{|\mathcal{S}|} p(\mathcal{S}) \mathbf{1}[i \in \mathcal{S}] \qquad (23) \\
&= \sum_{\mathcal{S}} \frac{C_{|\mathcal{S}|}}{|\mathcal{S}|} p(\mathcal{S}) \sum_{i=1}^{N} \mathbf{1}[i \in \mathcal{S}] \\
&= \sum_{\mathcal{S}} C_{|\mathcal{S}|} p(\mathcal{S}) \\
&\leq \max\{C_1, \cdots, C_N\},
\end{aligned}$$

where (23) follows from user and MPR channel symmetry. Thus, the maximum stable throughput of any MAC protocol is upper bounded by $\max\{C_1, \cdots, C_N\}$. $\square$

Since for some i, $C_i = \max\{C_1, \cdots, C_N\}$ we can conclude that a sum rate of C_i can be achieved by centralized scheduling of all sets consisting of i users. The above result is analogous to the result $\mathscr{S} \subseteq \mathscr{C}$ for the two user case.

The following theorem extends Theorem 3 to the symmetric $N > 2$ case.

Theorem 4: For the symmetric MPR channel, let

$$C_1 \geq \frac{C_2}{2} \geq \cdots \geq \frac{C_i}{i} \geq \cdots \geq \frac{C_N}{N}. \tag{24}$$

Then, the following are equivalent.

1) The reception probabilities satisfy

$$C_N = \max\{C_1, \cdots, C_N\}. \tag{25}$$

2) $\rho_{\text{ALOHA}}(1) = \rho_{\text{ALOHA}} = \rho = \max\{C_1, \cdots, C_N\}$.

Proof: Refer to the Appendix.

Theorem 4 shows that there is a regime for which ALOHA with transmission probability one *i.e.,* dumb scheduling is optimal amongst all MAC protocols. Equation (24) is equivalent to the condition that the probability of packet success per user decreases as interference increases. Equation (25) ensures that the expected number of successful receptions is maximized when all users transmit.

Note that for the orthogonal channel case, $C_i = iC_1$ and obviously dumb scheduling is optimal. As in the two user case, we see that dumb scheduling is optimal for a much larger class of symmetric MPR channels as specified by (25) and (24).

It is interesting to compare our results with those of [18] in which the problem of scheduling transmissions for the downlink of a multiple antenna cellular system is considered. Viswanath, Tse and Laroia show that from an information theoretic point of view, a good strategy for the base station is to employ "dumb" antennas (in the sense of not doing any signal processing other than that in a single antenna system) and implement "smart" scheduling (in the sense of scheduling users who have the best channel at that time). Thus, they show that more resources should be allocated to scheduling than to the physical layer for the downlink. Our problem is in some sense a conceptual dual of the downlink problem. Our results apply to the uplink of a multiple antenna cellular system and since we wish to address source burstiness, we choose the framework of random access. In contrast to [18], our results show the trade-off involved in allocation of resources to the MAC and the PHY layer.

Apart from the symmetric case, we cannot say whether such a result would carry over to the finite user case $(N > 2)$. However, we can still interpolate our two user results for systems with $N > 2$ by orthogonalizing all users into groups of two and implementing optimal scheduling or ALOHA or dumb scheduling depending on the level of MPR capability available in each group. Though this technique is suboptimal, it shows how a trade off between MAC layer complexity and physical layer complexity can be achieved. If the physical layer is strong enough to orthogonalize users into groups of two and guarantee that (19) holds for every group, then the MAC layer is not needed from a stability viewpoint.

B. Sufficient condition for the Asymmetric case, $N > 2$

Deriving stability conditions for the asymmetric $N > 2$ case is quite hard even for the collision channel model. Nonetheless, for a fixed transmission probability vector ($\mathbf{p}$), Szpankowski [5] gave a sufficient and necessary condition for stability of the ALOHA system with the collision channel model for the $N > 2$ case. In this section we restrict ourselves to finding sufficient conditions for stability for the general reception model for a fixed transmission probability vector ($\mathbf{p}$). The main ideas involved here are those of stochastic dominance and of constructing suitable dominating systems for which stability conditions are easier to determine. The way to construct such dominant systems is to assume that some of the queues in the system continue to transmit interfering dummy packets even when they are empty. Because of the dominance, sufficient stability conditions for the dominant system are enough for the original system as well. For the collision channel, such systems are known to stochastically dominate the original ALOHA system [3].

Let $\mathcal{P} = (\mathcal{V},\mathcal{U})$ be a partition of $\mathcal{M} \triangleq \{1, 2, \cdots, N\}$ such that users in $\mathcal{V} \neq \mathcal{M}$ behave just like those in the original ALOHA system while those in $\mathcal{U}$ continue to transmit dummy packets even when their queues are empty. We call users in $\mathcal{U}$ persistent and those in $\mathcal{V}$ nonpersistent. For a partition $\mathcal{P}$ defined above, let $\Theta^{\mathcal{P}}$ denote the ALOHA system where users behave as specified by $\mathcal{P}$. Further, let $\overline{\mathbf{Q}}^t_{\mathcal{P}} = (\overline{\mathbf{Q}}^t_{\mathcal{V}}, \overline{\mathbf{Q}}^t_{\mathcal{U}})$ denote the queue lengths in $\Theta^{\mathcal{P}}$.

We note that the marginal reception probabilities given by (2) are not enough to characterize the probability transition matrix for $\mathbf{Q}^t$. However, we find that the marginal probabilities given by (2) are enough to find sufficient conditions for stability of $\mathbf{Q}^t$ even for $N > 2$. We conjecture that the marginal probabilities of success are sufficient to completely characterize the stability region for a fixed transmission probability. For a slotted ALOHA system with set of (nonpersistent) users $\mathcal{M}$, we denote the set of marginal probabilities of success of all the users by $\mathbf{q}^{\mathcal{M}}$. More precisely, if $q^{\mathcal{M}}_{i|\mathcal{S}} \triangleq q_{i|\mathcal{S}}$, where $q_{i|\mathcal{S}}$ is defined by (2) then,

$$\mathbf{q}^{\mathcal{M}} = \{q^{\mathcal{M}}_{i|\mathcal{S}} : i \in \mathcal{S}, \mathcal{S} \subseteq \mathcal{M}\}. \tag{26}$$

We also assume that the reception probabilities ($\mathbf{q}^{\mathcal{M}}$) permit $\Theta^{\mathcal{P}}$ to stochastically dominate the original system.

The point to note is that the $|\mathcal{V}| < N$ dimensional process $\overline{\mathbf{Q}}^t_{\mathcal{V}}$ is also a Markov Chain which mimics the original ALOHA system [5] except with *modified* reception probabilities

($\mathbf{q}^{\mathcal{V}} = \{q^{\mathcal{V}}_{i|\mathcal{S}}, i \in \mathcal{S}, \mathcal{S} \subseteq \mathcal{V}\}$). Thus, we can use induction arguments to establish its stability. More precisely, for any $\mathcal{V}' \subseteq \mathcal{V}$ and $i \in \mathcal{V}'$, the modified reception probabilities for the smaller ALOHA system consisting of the stand alone non-persistent set $\mathcal{V}$ become

$$q^{\mathcal{V}}_{i|\mathcal{V}'} = \sum_{\mathcal{T} \subseteq \mathcal{U}} \left(\prod_{j \in \mathcal{T}} p_j \prod_{k \in \mathcal{U} \setminus \mathcal{T}} \overline{p}_k \right) q^{\mathcal{M}}_{i|\mathcal{V}' \cup \mathcal{T}}. \tag{27}$$

Now, suppose that the Markov Chain $\overline{\mathbf{Q}}^t_{\mathcal{V}}$ is stationary and ergodic. We denote the stationary version of queue lengths in the non-persistent set by $\overline{\mathbf{Q}}_{\mathcal{V}}$. If we initialize $\overline{\mathbf{Q}}^t_{\mathcal{V}}$ with its stationary distribution, the departure process from jth users' queue in $\Theta^{\mathcal{P}}$ is also stationary and ergodic. Let $\mathcal{V} = \{v_1, v_2, \cdots, v_{|\mathcal{V}|}\}$ and define $\mathcal{X}(\mathbf{Q}) = (\mathbf{1}[Q_1 > 0], \mathbf{1}[Q_2 > 0], \cdots, \mathbf{1}[Q_{|\mathbf{Q}|} > 0])$, where $\mathbf{1}[\cdot]$ is the indicator function. Also, for $\mathbf{z}_{\mathcal{V}} = (z_1, z_2, \cdots, z_{|\mathcal{V}|}) \in \{0,1\}^{|\mathcal{V}|}$, define $\mathcal{C}(\mathcal{V}, \mathbf{z}_{\mathcal{V}}) = \{v_i : z_i = 1, i \leq |\mathcal{V}|\}$. For $j \in \mathcal{U}$, let $P^j_{\Theta^{\mathcal{P}}}$ be the probability of success of the jth user in $\Theta^{\mathcal{P}}$ in the stationary version constructed above. Then, we have,

$$P^j_{\Theta^{\mathcal{P}}} = p_j \left[\sum_{\mathbf{z}_{\mathcal{V}} \in \{0,1\}^{|\mathcal{V}|}} \Pr\{\mathcal{X}(\overline{\mathbf{Q}}_{\mathcal{V}}) = \mathbf{z}_{\mathcal{V}}\} \left(\sum_{\mathcal{B} \subseteq \mathcal{U} \cup \mathcal{C}(\mathcal{V}, \mathbf{z}_{\mathcal{V}}) \setminus j} \prod_{k \in \mathcal{B}} p_k \prod_{l \in \mathcal{U} \cup \mathcal{C}(\mathcal{V}, \mathbf{z}_{\mathcal{V}}) \setminus (j \cup \mathcal{B})} \overline{p}_l q_{j|j \cup \mathcal{B}} \right) \right]. \tag{28}$$

Now define a region $\mathcal{R}(\mathbf{q}^{\mathcal{M}})$ recursively as,

$$\mathcal{R}(\mathbf{q}^{\mathcal{M}}) \triangleq \bigcup_{\mathcal{P}} \{\boldsymbol{\lambda}_{\mathcal{M}} = \{\lambda_i\}_{i \in \mathcal{M}} : \lambda_k < P^k_{\Theta^{\mathcal{P}}} \ \forall\, k \in \mathcal{U}, \text{ and } \boldsymbol{\lambda}_{\mathcal{V}} \in \mathcal{R}(\mathbf{q}^{\mathcal{V}})\}, \tag{29}$$

with $\mathcal{R}(\mathbf{q}^{\{i\}}) \triangleq \{\lambda_i < p_i q^{\{i\}}_{i|i}\}$.

Now, we claim the sufficient condition for stability in the form of this theorem.

Theorem 5: Under conditions of stochastic dominance of $\Theta^{\mathcal{P}}$ over the original ALOHA system, if $\boldsymbol{\lambda}_{\mathcal{M}} \in \mathcal{R}(\mathbf{q}^{\mathcal{M}})$, then the ALOHA system is stable. In other words, $\mathcal{R}(\mathbf{q}^{\mathcal{M}}) \subseteq \mathscr{S}_{\text{ALOHA}}$.

Proof: Refer to the Appendix.

The reasoning behind why $\boldsymbol{\lambda}_{\mathcal{M}} \in \mathcal{R}(\mathbf{q}^{\mathcal{M}})$ is sufficient for stability is quite simple; for a particular partition $\mathcal{P}$, $\boldsymbol{\lambda}_{\mathcal{V}} \in \mathcal{R}(\mathbf{q}^{\mathcal{V}})$ is sufficient for stability (by induction arguments) of the Markov Chain consisting of the non-persistent set and this makes the departure process for queues in the persistent set stationary and ergodic. Then, $\lambda_k < P^k_{\Theta^{\mathcal{P}}} \ \forall\, k \in \mathcal{U}$ is sufficient for stability of persistent queues by Loynes theorem. Thus, $\Theta^{\mathcal{P}}$ is stable and by virtue of stochastic dominance, the original system is stable.

As an example, consider the asymmetric orthogonal channel case. The inner bound for the stability region given by Theorem 5 for this case is actually tight. This follows since for orthogonal channels we have,

$$q_{i|\mathcal{S}}^{\mathcal{M}} = q_{i|i}^{\{i\}} \qquad \forall i \in \mathcal{S}, \forall \ \mathcal{S} \subseteq \mathcal{M}. \tag{30}$$

It is easy to check that the reception probabilities satisfying (30) are sufficient for stochastic dominance. Further, it can also be easily seen that the region $\mathcal{R}(\mathbf{q}^{\mathcal{M}})$ for a fixed transmission probability vector $\mathbf{p}$ is given by

$$\mathcal{R}(\mathbf{q}^{\mathcal{M}}) = \{\boldsymbol{\lambda}_{\mathcal{M}} : \lambda_i < p_i q_{i|i}^{\{i\}} \quad \forall \ i \in \mathcal{M}\}. \tag{31}$$

Clearly, $\mathbf{p} = \mathbf{1}$ is the best policy in such a case. Similarly, for the case of orthogonal doublets the sufficient condition given by Theorem 5 is tight since in this case, $\mathcal{R}(\mathbf{q}^{\{1,2\}}) = \mathscr{S}_{\text{ALOHA}}$ for all the orthogonal doublets. However, the sufficient condition given by Theorem 5 is too difficult if not impossible to evaluate in practice in general. This is because evaluating the stationary distribution of the queues for $N > 2$ with arbitrary input distributions in closed form is an unsolved problem as observed in [5], [2], [6], [19].

VII. MPR THROUGH MULTIPLE ANTENNAS: AN EXAMPLE

A. Two User case

To get more insights into the analytical results in section V, we now apply our results to a two-user scenario to compare different receiver front-ends.

We consider two users, each communicating with a central base station that employs a linear array of M antennas. The two users use slotted ALOHA as the MAC. We assume that the slots are synchronized. The two users are located relatively far away from the base station at fixed angular positions $\boldsymbol{\theta} = [\theta_1, \theta_2]$ with respect to the array normal. We assume that most of the energy from user transmissions is received from a planar wavefront arriving at the angle $\boldsymbol{\theta}$. Under these assumptions, we can describe the received signal at the base station $\mathbf{y}$ as

$$\mathbf{y} = \mathbf{V}(\boldsymbol{\theta})\mathbf{H}\,\mathbf{s} + \mathbf{n}, \tag{32}$$

where $\mathbf{V}(\boldsymbol{\theta})$ is a Vandermonde matrix of array responses, $\mathbf{H} =$ diag$[h_1, h_2]$, is a diagonal matrix of channel (flat) fading for the two users, $\mathbf{s} = [s_1, s_2]^T$, is a vector of users' transmitted symbols and $\mathbf{n} \sim \mathcal{CN}(0, \mathbf{I}_M)$, is additive white gaussian noise.

We also assume that the channel fading is slow and it is independent for the two users and also is i.i.d. from slot to slot. For our numerical results, we assume Rayleigh fading with zero mean and covariance matrix $\Sigma = \text{ diag}[\sigma^2_{h_1}, \sigma^2_{h_2}]$. User symbols ($s_i,\ i = 1, 2$) are independent of each other and the channel fading with $\mathbb{E}(|s_i|^2) = 1,\ i = 1, 2$. We also assume noncoherent receiver operation *i.e.*, the base station does not know the channel realization, when it implements the front-end. We consider the effect of coherent receiver operation and knowledge of queue statistics at the base station in [20]. We represent the front-end processing by $\mathbf{F}$ (ith row of $\mathbf{F}$ is the set of beamforming weights for the ith user) as follows:

$$\begin{aligned} \mathbf{z} = \mathbf{F}\mathbf{y} &= \overbrace{\mathbf{F}\mathbf{V}(\boldsymbol{\theta})\mathbf{H}}^{\mathbf{R}}\mathbf{s} + \mathbf{F}\mathbf{n} \\ &= \mathbf{R}\mathbf{s} + \mathbf{w}. \end{aligned} \tag{33}$$

The most important assumption we make is that of the SINR threshold model for packet success *i.e.*, a packet is successfully received and decoded for user i if

$$\mathbb{E}(\text{SINR}_i|\mathbf{H}) = \frac{|r_{ii}|^2}{|r_{i\bar{i}}|^2 + \mathbb{E}(|w_i|^2)} \geq \beta. \tag{34}$$

where the expectation is taken over user symbols and noise. In the above, $\bar{i} = \{1, 2\} \setminus \{i\}$ and β is a threshold which depends on the quality of service requirement. Under the SINR threshold model, the $\mathbf{q}$ vector of packet success probabilities for a particular $\mathbf{F}$ can be found as

$$\begin{aligned} q_{i|\{i\}} &= \text{Pr}\left\{ \frac{|r_{ii}|^2}{\mathbb{E}(|w_i|^2)} > \beta \right\}, \\ q_{i|\{1,2\}} &= \text{Pr}\left\{ \frac{|r_{ii}|^2}{|r_{i\bar{i}}|^2 + \mathbb{E}(|w_i|^2)} > \beta \right\}. \end{aligned}$$

The explicit computation of $\mathbf{q}$ is provided in the appendix. We now consider the performance of three different front-ends for the above system:

1) *De-correlating or Zero-forcing* (ZF): $\mathbf{F}$ is the pseudo-inverse of $\mathbf{V}(\boldsymbol{\theta})$.
2) *Matched Filter* (MF): $\mathbf{F} = \mathbf{V}^H(\boldsymbol{\theta})$.
3) *pseudo-MMSE* (pMMSE): For this receiver, $\mathbf{F} = \text{diag}[\sigma^2_{h_1}, \sigma^2_{h_2}]\mathbf{V}^H(\boldsymbol{\theta})\mathbf{R}^{-1}_{yy}$, where $\mathbf{R}_{yy}$ is the correlation matrix of $\mathbf{y}$ assuming both users transmit. Note that the perfect MMSE receiver needs to know which users are transmitting and the channel realizations in order to find the optimal weights.

Using the stability region as a figure of merit, we can now compare the stability regions of these front-ends in various situations of interest.

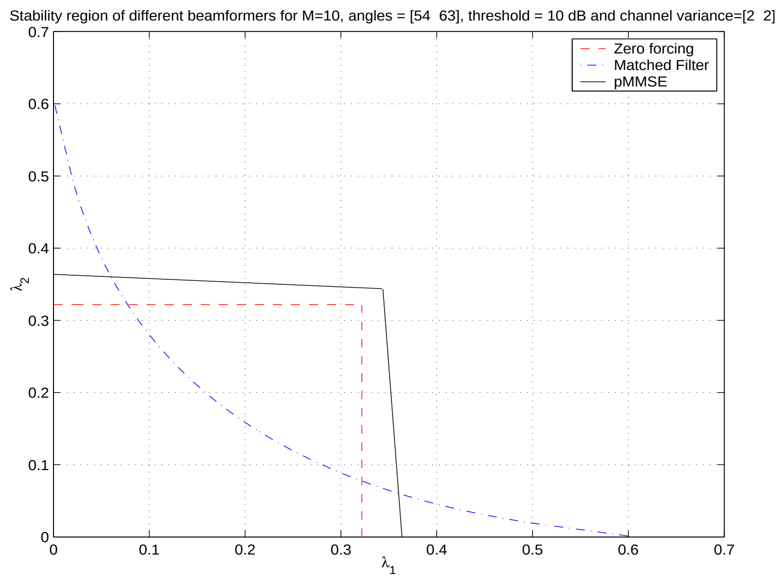


Fig. 8. $M = 10, \boldsymbol{\theta} = [54, 63]$, Threshold $= 10dB$, channel gain $= 3dB$

1) Symmetric Case: In this case, the channels for the two users are symmetric *i.e.* $\sigma^2_{h_1} = \sigma^2_{h_2}$. In Figure 8, we see stability regions for the three different front-ends when the two users are relatively close, $\boldsymbol{\theta} = [54, 63]°$. We observe that in this rather pessimistic scenario, when one of the users demands a very low rate (close to the axes) the MF performs better than the ZF and pMMSE. This is not surprising since the MF is optimal if only one user transmits; the ZF suffers from noise enhancement whereas the pMMSE assumes that both users transmit in every slot. On the other hand the ZF and pMMSE perform much better than the MF when both users demand an equal rate since in that case both the ZF and pMMSE suppress the interference from the other user better than the MF. We also note that the stability region of the ZF is a rectangle since, the ZF decouples the two users' signals.

2) Asymmetric Case: Figure 9 shows the situation when the second user has a very good channel as compared to the first and the users are almost collinear, $\boldsymbol{\theta} = [54, 58]°$. We see a near-far effect with the ZF and pMMSE front-ends, whereas the MF performs very well. It is not surprising since the MF does not really attempt to null out the other user while the ZF and

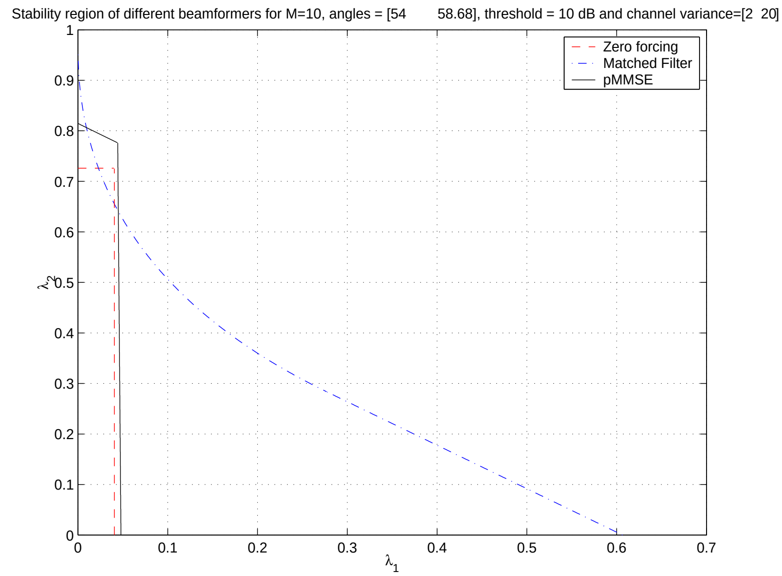


Fig. 9. $M = 10, \boldsymbol{\theta} = [54, 58]$, Threshold $= 10dB$, channel gains $= [3, 13]dB$

pMMSE do that. Because of the angular proximity, the ZF and pMMSE suffer. We also note that the stability region of the pMMSE contains the stability region of the ZF receiver in Figures 8 and 9.

VIII. Delay performance of ALOHA

Now, we consider characterizing delay in slotted ALOHA systems with multipacket reception. Sidi and Segall [21] analyzed delay in buffered ALOHA type systems and found the exact average delay in a two user system with symmetric arrival rates and transmission probabilities under a collision channel. They also found optimal transmission probabilities to minimize delay. Further, Nain [22] calculated the exact delay in the two user case for asymmetric arrivals and transmission probabilities assuming a collision channel. The technique used to find delay involves solving a functional equation in the generating function of the joint stationary queue length distribution. This functional equation can be solved by formulating a Riemann-Hilbert boundary value problem [23], [24]. It is indeed quite surprising that there are no results on the "exact" delay of ALOHA for this queueing model apart from these two. Takagi and Kleinrock [25] use a similar approach to find average delay in a two-user buffered CSMA/CD system with a collision channel. There is also a line of work that computes bounds on average delay for

$N > 2$ for the collision channel ([3], [26], [27], [19]) and for a more general symmetric MPR model [10]. There are quite a few other results on delay of ALOHA but they are for different queueing models *viz.*, infinite user single buffer, finite user single buffer. These models do not quite capture the interdependence amongst the queues and its effect on delay. The limited results found suggest that characterizing delay in ALOHA systems is a nontrivial task.

In this section, we characterize delay for a subclass of MPR channels *viz.*, capture channels. In a capture channel, there is a chance that *at most* one user has a successful packet transmission even if many users transmit in that slot. In some sense, it is an elementary generalization of the collision channel with probabilistic receptions. We focus our attention on the two user symmetric ALOHA system. We assume that every user has an infinite buffer in which he can store arriving and backlogged packets. The arrivals to the ith user are i.i.d. Bernoulli(λ_i) in every slot. The arrivals are independent across users. The reception model is like the one in the preceding sections.

By definition, for a capture channel, $q_{\{1,2\}|\{1,2\}} = 0$. Let $q_{1|\{1\}} = q_{2|\{2\}} = a$ and $q_{1|\{1,2\}} = q_{2|\{1,2\}} = b$. Further, we assume $a \geq b$ (to use results of the previous sections). Note that the capture model implies that $a \leq 1$ and $b \leq 0.5$. Also, let $\lambda_1 = \lambda_2 = r$ and p be the transmission probability of both users.

Theorem 6: Let D be the average delay for either user in the symmetric capture channel. If the system is stable *i.e.*, $r < pa + p^2(b-a)$,

$$D = \frac{1}{a}\left[\frac{a(1-r) + p(b-a)(1-r/2)}{pa + p^2(b-a) - r}\right]. \tag{35}$$

Proof: Refer to the Appendix.

From (35), we observe that the delay is a decreasing function of r, as expected.

Next, we look at the problem of optimizing the transmission probability (p^*) to minimize the average delay. We find that as soon as there is capture capability, the optimal transmission probability is one for a set of arrival rates of the form $[0, r^*]$ with $r^* > 0$. Thus, dumb scheduling is delay optimal in the class of ALOHA protocols with fixed transmission probability for small arrival rates.

Lemma 3: Let p^* be the optimal transmission probability for minimizing delay in the capture channel. Then,

$$p^* = \begin{cases} 1 & r \in [0, r^*] \\ \frac{a(1-r)-\sqrt{0.5r}\sqrt{2(a-b)(1-r/2)^2-a^2(1-r)}}{(a-b)(1-r/2)} & r \in (r^*, r_{max}), \end{cases} \tag{36}$$

where,

$$r^* = \begin{cases} 1 - \left(\frac{a+b}{2} - \frac{ab}{a-b}\right) - \sqrt{1-(a-b)+\left(\frac{a+b}{2} - \frac{ab}{a-b}\right)^2} & 0 \leq b < a/2 \\ b & a/2 \leq b \leq \min\{0.5, a\}, \end{cases} \tag{37}$$

and

$$r_{max} = \begin{cases} \frac{a^2}{4(a-b)} & 0 \leq b < a/2 \\ b & a/2 \leq b \leq \min\{0.5, a\}. \end{cases} \tag{38}$$

Proof: Refer to the Appendix. □

Lemma 3 gives p^* explicitly in terms of the capture channel parameters and the arrival rate r. As a direct consequence of Lemma 3 we have the following theorem.

Theorem 7: For the capture channel with $a > 0, b > 0$, the optimal transmission probabilities can take only two possible forms *viz.*,

1) If $b < a/2$, then the optimal transmission probability is one for a non-empty proper subset of all stable rates of the form $[0, r^*]$ with $0 < r^* < r_{max}$.
2) If $b \geq a/2$, then the optimal transmission probability is one for all stable arrival rates.

Proof: For a fixed $a > 0$, from Lemma 3 note that,

$$r^*|_{b=0} = 0.$$

It can be shown that r^* is a strictly increasing function of b for a fixed value of $a > 0$. Thus, as soon as we have capture ($b > 0$), $r^* > 0$ and there is a set of rates $[0, r^*]$ for which $p^* = 1$ is the best policy for minimizing delay. As long as $b < a/2$, we have $r^* < r_{max}$. On the other hand when $b \in [a/2, \min\{0.5, a\})$, from Lemma 3

$$(r^* = r_{max})|_{b \in [a/2, \min\{0.5, a\})} = b, \tag{39}$$

and so $p^* = 1$ is delay optimal for any rate which is stabilizable. □

Note that for $b < a/2$, $r^* < r_{max}$, and there is a set of rates for which the optimal transmission probability p^* is still a function of the arrival rate (r). Thus, $b = a/2$ also happens to be the point where the optimal transmission probability ceases to be a function of the arrival rate. We refer to r^* as the *critical* rate since rates below r^* are delay optimized by dumb scheduling. In section V, we have already shown that r_{max} is the maximum stable arrival rate for the capture model.

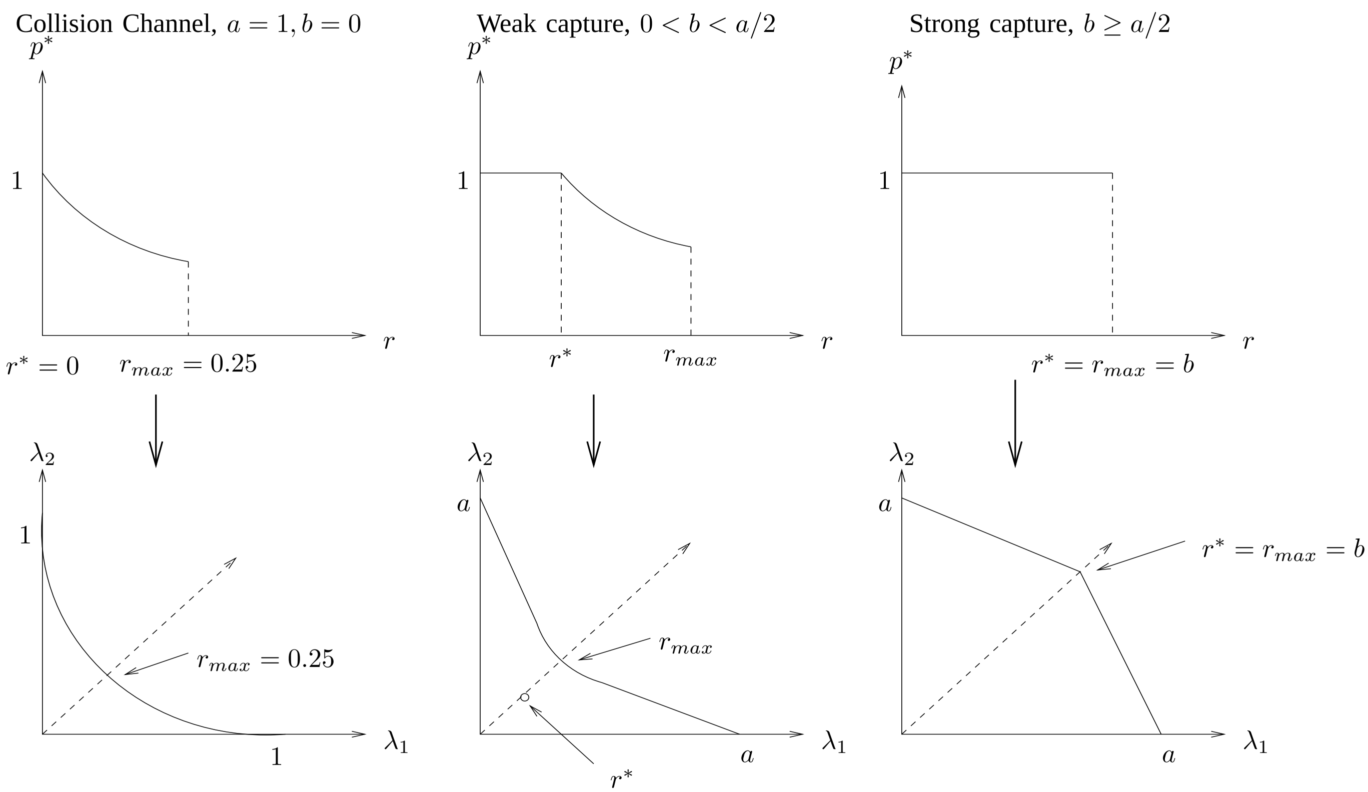


Fig. 10. Generic optimal transmission probabilities for capture channels.

Figure 10 shows the generic optimal transmission probabilities as a function of the capture channel parameters. It is interesting to compare the structure of the stability region along the equal rate line with the optimal transmission probability for different capture models. Note that $b = a/2$ is also the point at which dumb scheduling is optimal from a stability viewpoint. Thus, dumb scheduling is optimal from both delay and stability viewpoints when $b \geq a/2$.

Figure 11 shows the set of transmission probabilities that stabilize the ALOHA system for

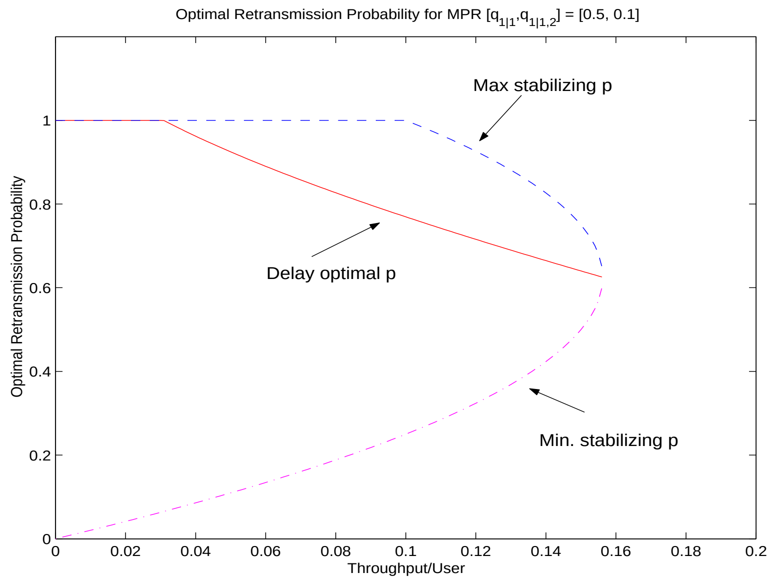


Fig. 11. The optimal and stabilizing transmission probabilities for $a = 0.5$, $b = 0.1$

different arrival rates in a weak capture ($b < a/2$) case. The maximum and minimum stabilizing transmission probabilities are the solution to the equation $p^2(b - a) + pa - r = 0$ and thus form a parabola which is truncated since the maximum transmission probability can be at most one. The point at which the maximum and minimum transmission probabilities coincide corresponds to the maximum stable arrival rate r_{max}. The delay optimal transmission probability lies in the feasible region in the interior of the parabola.

Now, we look at the delay results in various situations of interest.

A. Maximum stable throughput and Critical rate

Figures 12 and 13 compare the critical rate with maximum stable arrival rate for all possible capture scenarios. We see a phase transition here that occurs at the point $b = a/2$. As long as $b < a/2$, dumb scheduling is only optimal for a subset of the stabilizable rates. On the other hand as soon as $b \geq a/2$, dumb scheduling is optimal for all stabilizable rates. Note that all rates below the solid curve are delay optimized by dumb scheduling.

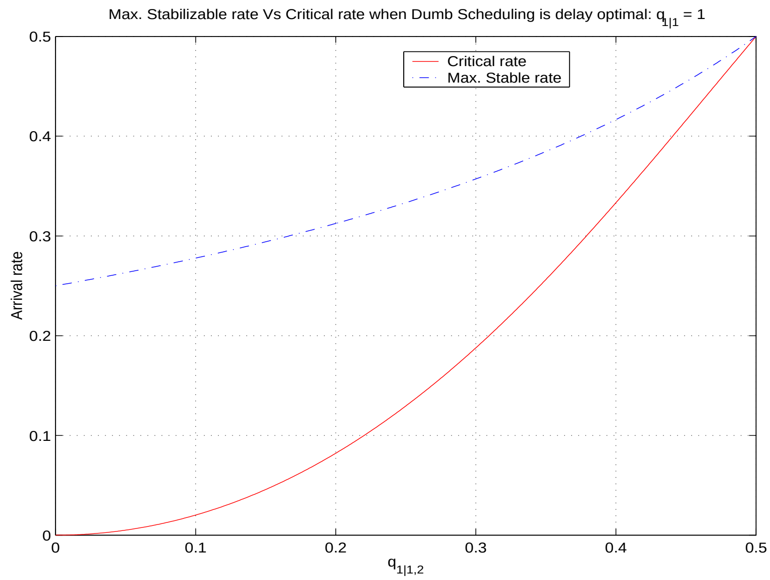


Fig. 12. Maximum stable throughput vs critical rate $q_{1|\{1\}} = a = 1$

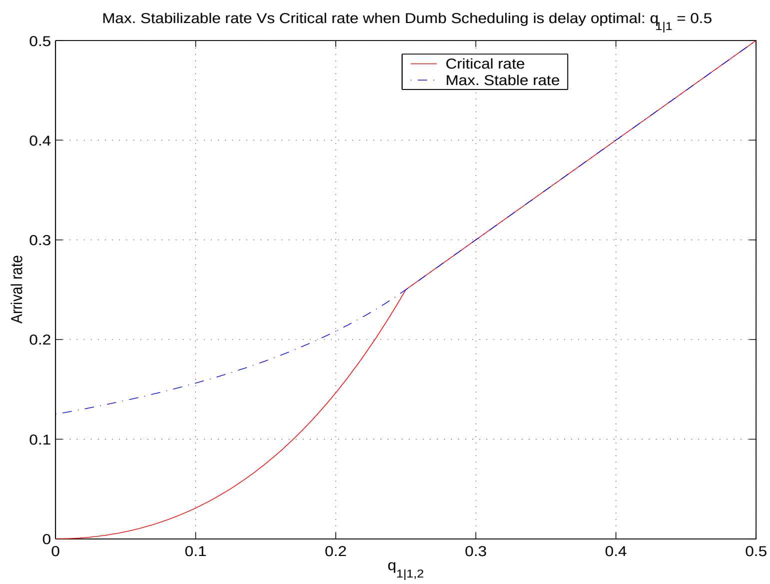


Fig. 13. Maximum stable throughput vs critical rate $q_{1|\{1\}} = a = 0.5$

B. Delay comparison of different capture channels

Figure 14 compares the minimal delay for three capture scenarios. In this case, we increase a and decrease b progressively. It can be seen that at low arrival rates the capture model with

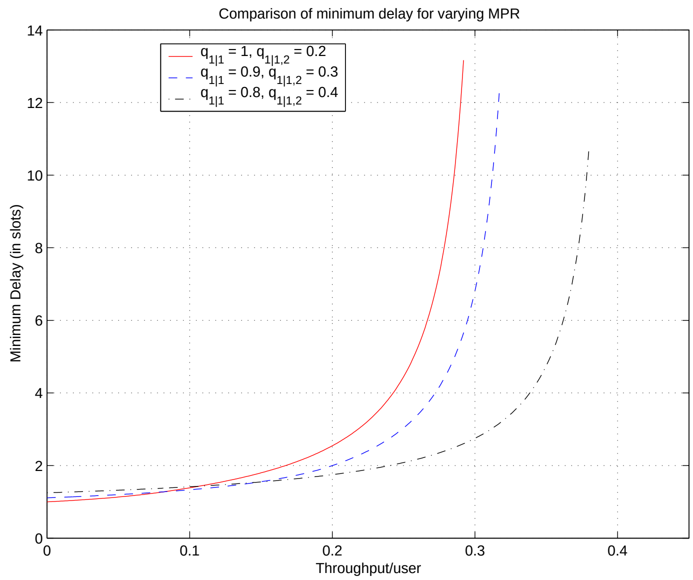


Fig. 14. Comparison of delay for various capture scenarios, $q_{1|\{1\}} = a = 1, 0.9, 0.8$, $q_{1|\{1,2\}} = b = 0.2, 0.3, 0.4$

$a = 1$, $b = 0.2$ is marginally better that the other capture models. At higher arrival rates, the capture model with $a = 0.8$, $b = 0.4$ is significantly better than the others. Thus, it seems that for minimizing delay, "multiuser" receiver design is much better than the omnipresent "single user" designs.

Figure 15 compares the minimal delay in collision channel $a = 1$, $b = 0$ with the delay in strong capture scenarios. It illustrates the significant average delay reduction that can be achieved with the strongest capture model $a = 1$, $b = 0.5$. We also note that the minimal delay in this strong capture model ($a = 1$, $b = 0.5$) is quite close to one for arrival rates until ~ 0.25. Since the average delay is lower bounded by one, this suggests that ALOHA is quite close to optimal for a large class of arrival rates for strong capture models.

C. Delay comparison with fixed "a"(or "b")

Figures 16 and 17 show minimal delay as a function of the arrival rate for fixed $q_{1|\{1\}} = a$ and fixed $q_{1|\{1,2\}} = b$ respectively. In Figure 16 the curves are far apart as compared to those in Figure 17. The figures show that the delay is much more sensitive to changes in b than a.

At this point it is not clear what would happen if we had a stronger reception model than the capture model we have considered in this work. First of all, the technique used to find

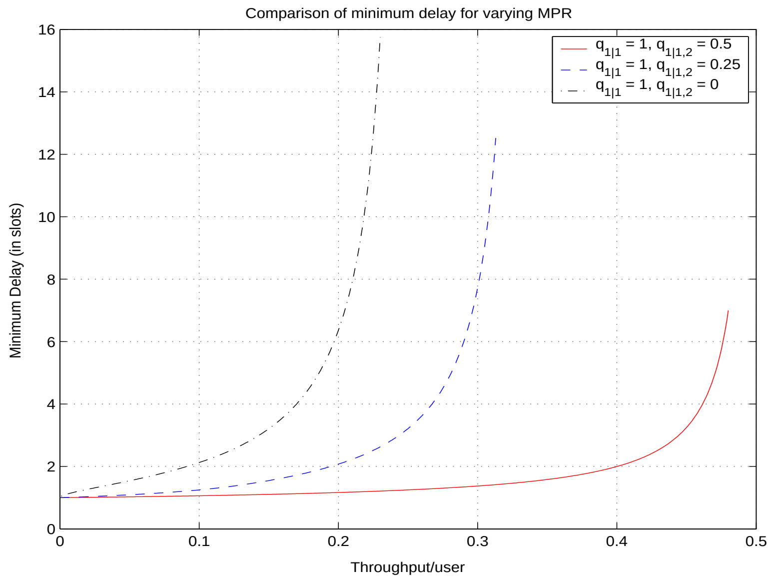


Fig. 15. Comparison of delay for capture scenarios with the collision channel, $q_{1|\{1\}} = a = 1, 1, 1$, $q_{1|\{1,2\}} = b = 0.5, 0.25, 0$

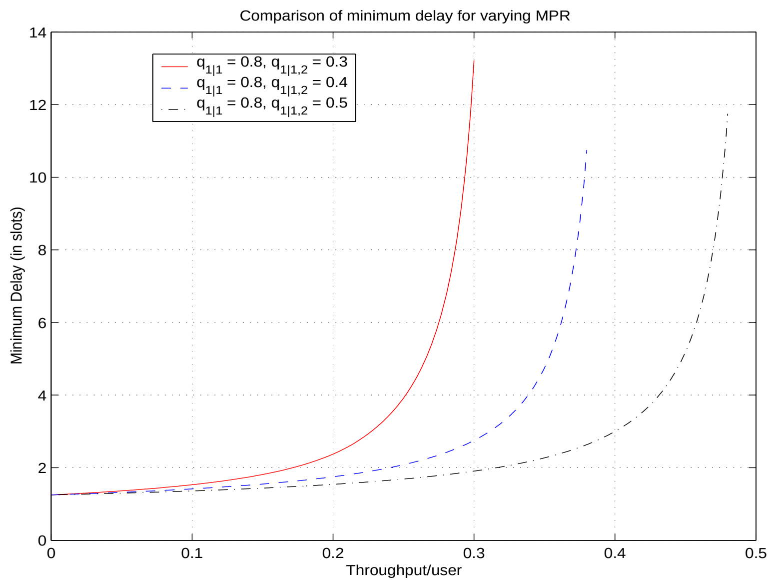


Fig. 16. Comparison of delay for various capture scenarios, $q_{1|\{1\}} = a = 0.8, 0.8, 0.8$, $q_{1|\{1,2\}} = b = 0.3, 0.4, 0.5$

the average queue length fails as terms corresponding to probability of success of both users *simultaneously* lead to some complications. However, we conjecture that even with a stronger (symmetric) reception model *i.e.* $2q_{1|\{1,2\}} > 1$, $p = 1$ would minimize the delay. The intuition

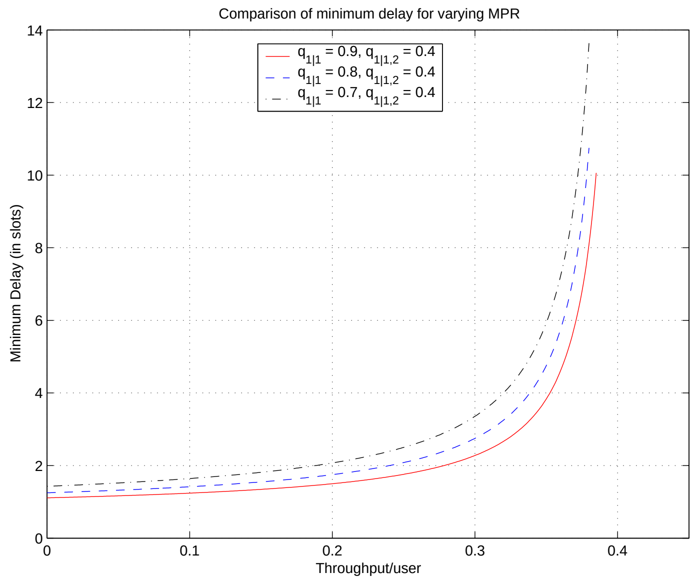


Fig. 17. Comparison of delay for various capture scenarios, $q_{1|\{1\}} = a = 0.9, 0.8, 0.7$, $q_{1|\{1,2\}} = b = 0.4, 0.4, 0.4$

behind this belief is that there is no reason for the users to hold back transmissions in the presence of a stronger reception model.

We also suspect that the delay results that we have will be upper bounds on delay of MPR models which have the same marginal reception probabilities as the capture model we consider *i.e.* for MPR models with $q_{1|\{1\}} = a$ and $q_{1|\{1,2\}} = b$ with $q_{\{1,2\}|\{1,2\}} > 0$.

IX. Conclusions

In this paper, we considered the problem of characterizing stability and delay behavior of slotted ALOHA for Multipacket reception models intended to capture the behavior of an improved physical layer. It was shown that the stability region of slotted ALOHA for the two user case has only four possible forms and that the stability region shows a phase transition from non-convexity to convexity as the MPR capability increases. After the phase transition, slotted ALOHA was shown to be optimal amongst all MAC protocols from the point of view of stability. It was also found that when slotted ALOHA is optimal, ALOHA with a transmission probability one (referred to as Dumb Scheduling) is optimal. This suggests that after a certain level of MPR capability is provided by the PHY layer, MAC layer design is simple; dumb scheduling is the best policy. Further, it was shown that dumb scheduling is also optimal for the $N > 2$ case for the symmetric MPR model in a wide range of MPR regimes. Generalizing the stability region results to the finite user case seems a rather non-trivial task and although we have been able to provide some sufficient conditions for stability, they are difficult to compute. Exact expressions for delay minimizing transmission probabilities and the exact average delay of slotted ALOHA for a subclass of MPR models (capture channel) for the two user case were provided. It was also shown that dumb scheduling is optimal from a delay *and* stability viewpoint for all stable arrival rates in a certain capture regime and that dumb scheduling is always delay optimal for a subset of stable arrival rates once capture sets in. Our results present a clear case for the so called "cross layer" approach, (where a combination of layers (here PHY and MAC) are designed jointly to optimize network performance) by quantifying the gains that can be achieved through an optimal design of the MAC based on an accurate model of the PHY layer.

APPENDIX

A. Proof of Theorem 1

We provide the main steps of the proof.

(a) *if* part: The proof relies on randomized scheduling. Since all users have infinite packets to send, in every slot choose a set $\mathcal{S}$ of users to transmit with probability $p(\mathcal{S})$. By the strong law of large numbers the success rate for the ith user is

$$\sum_{\mathcal{S}\subseteq\mathcal{M}} q_{i|\mathcal{S}}\, p(\mathcal{S}) \quad a.s. \tag{40}$$

Therefore, every rate $\boldsymbol{\lambda}$ satisfying (8) for all i is achievable.

(b) *only if* part: Let $\boldsymbol{\lambda}$ be achievable. Then, there exists a scheduling policy S^* such that

$$\liminf_{T\to\infty} \frac{1}{T}\sum_{t=0}^{T-1} \mathbf{1}(i \in \mathcal{R}_{S^*}(t)) \geq \lambda_i \qquad \forall i \quad a.s.$$

We will show that there exists a probability measure $\{p(\mathcal{S}) : \mathcal{S}\subseteq\mathcal{M}\}$ such that for any $\epsilon > 0$,

$$\lambda_i - \epsilon \leq \sum_{\mathcal{S}\subseteq\mathcal{M}} q_{i|\mathcal{S}}\, p(\mathcal{S}). \tag{41}$$

Now, if $\boldsymbol{\lambda}$ is achievable, then

$$\begin{aligned}
\lambda_i &\leq \mathbb{E}\left(\liminf_{T\to\infty}\frac{1}{T}\sum_{t=0}^{T-1}\mathbf{1}(i\in\mathcal{R}_{S^*}(t))\right) && (42)\\
&\leq \liminf_{T\to\infty}\frac{1}{T}\sum_{t=0}^{T-1}\mathbb{E}(\mathbf{1}(i\in\mathcal{R}_{S^*}(t))) \qquad \text{(Fatou's Lemma)} && (43)\\
&= \liminf_{T\to\infty}\frac{1}{T}\sum_{t=0}^{T-1}\ \Pr\{i\in\mathcal{R}_{S^*}(t)\}. && (44)
\end{aligned}$$

The last inequality implies that there exists a T such that for all i,

$$\lambda_i-\epsilon\leq\frac{1}{T}\sum_{t=0}^{T-1}\ \Pr\{i\in\mathcal{R}_{S^*}(t)\}. \tag{45}$$

Define a probability measure $\{p(\mathcal{S}):\mathcal{S}\subseteq\mathcal{M}\}$ by

$$p(\mathcal{S})=\frac{1}{T}\sum_{t=0}^{T-1}\ \Pr\{\mathcal{S}=\mathcal{S}_{S^*}(t)\}. \tag{46}$$

By simple manipulations, it follows that

$$\frac{1}{T}\sum_{t=0}^{T-1}\ \Pr\{i\in\mathcal{R}_{S^*}(t)\}=\sum_{\mathcal{S}\subseteq\mathcal{M}}q_{i|\mathcal{S}}p(\mathcal{S}). \tag{47}$$

Substituting (47) in (45) gives us (41). Since ϵ is arbitrary, we are done. $\Box$

B. Proof of Theorem 2

Let $\mathbb{S}$ denote the power set of the set $\mathcal{M}=\{1,2,\cdots,N\}$. Let $\{p(\mathcal{S})\}_{\mathcal{S}\in\mathbb{S}}$ be a measure on $\mathbb{S}$. In order to prove Theorem 2, we use the following proposition.

Proposition 1: If the arrival rate $\lambda=(\lambda_1,\lambda_2,\cdots,\lambda_N)$ is stable (with any MAC protocol) then $\forall\ \epsilon>0,\exists$ a $\{p(\mathcal{S})\}_{\mathcal{S}\in\mathbb{S}}$ such that

$$\lambda_i-\epsilon\mathbf{1}(\lambda_i>0)\leq\sum_{\mathcal{S}\in\mathbb{S}}p(\mathcal{S})q_{i|\mathcal{S}}\quad\forall\ i\in\mathcal{M}. \tag{48}$$

Equivalently, $\mathscr{S}\subseteq\mathscr{C}$.

Before we prove the proposition we need a few intermediate results. We begin by focusing our attention on any specific MAC. So, let $S(t)$ denote the set of users that the MAC protocol allows to transmit in time slot t. We implicitly assume that the MAC protocol knows which users have packets to send. Note that $S(t)$ could be deterministic or random depending on the specific MAC protocol. It could also be a function of the past history of transmissions and/or packet success outcomes and/or arrivals. As before, let Q_i^t denote the queue length of the ith user at the beginning of time slot t. Further, let β_i^t denote the number of packets which arrive at the ith users buffer at the beginning of time slot t. By assumption $\{\beta_i^t\}_{t=0}^{\infty}$ is i.i.d. and $\mathbb{E}(\beta_i^t) = \lambda_i$. Let R_i^t denote the departure process from the ith users' queue *i.e.*,

$$R_i^t = \begin{cases} 1, & \text{if } i\text{th users' packet is successfully received in time slot t} \\ 0, & \text{otherwise}. \end{cases} \tag{49}$$

Then, we have,

Lemma 4: If the system is stable, then $\forall\ i \in \mathcal{M}$

$$\frac{1}{t}\mathbb{E}(Q_i^t) \to 0 \text{ as } t \to \infty.$$

If we assume that Lemma 4 is true, then $\forall\ \epsilon > 0$, we can find a $T > 0$ large enough such that $\forall\ i \in \mathcal{M}$,

$$\mathbb{E}\left(\frac{Q_i^T}{T}\right) < \epsilon. \tag{50}$$

Now, we define a measure $\{p(\mathcal{S})\}_{\mathcal{S}\in\mathbb{S}}$ as follows.

$$p(\mathcal{S}) = \frac{1}{T}\sum_{t=0}^{T-1} \text{Pr}\{S(t) = \mathcal{S}\}. \tag{51}$$

Lemma 5: Let $\{p(\mathcal{S})\}_{\mathcal{S}\in\mathbb{S}}$ and T be defined by equations (51) and (50). Then, $\forall\ i \in \mathcal{M}$ we have

$$\mathbb{E}\left(\frac{1}{T}\sum_{t=0}^{T-1} R_i^t\right) = \sum_{\mathcal{S}\in\mathbb{S}} q_{i|\mathcal{S}}\, p(\mathcal{S}).$$

Now we are in a position to prove Proposition 1.

Proof of Proposition 1: First we note that all the arriving packets till time T are either transmitted successfully or are queued up in the buffer. So, we have $\forall\ i \in \mathcal{M}$

$$\frac{1}{T}\sum_{t=0}^{T-1} R_i^t = \frac{1}{T}\sum_{t=0}^{T-1} \beta_i^t - \frac{1}{T}Q_i^T \quad \textit{a.s.}, \tag{52}$$

which implies $\forall\ i \in \mathcal{M}$

$$\mathbb{E}\left(\frac{1}{T}\sum_{t=0}^{T-1} R_i^t\right) = \mathbb{E}\left(\frac{1}{T}\sum_{t=0}^{T-1} \beta_i^t - \frac{1}{T}Q_i^T\right). \tag{53}$$

By Lemma 5 and (53) we have,

$$\sum_{\mathcal{S}\in\mathbb{S}} q_{i|\mathcal{S}}p(\mathcal{S}) = \mathbb{E}\left(\frac{1}{T}\sum_{t=0}^{T-1}\beta_i^t - \frac{1}{T}Q_i^T\right) \quad (54)$$

$$= \lambda_i - \mathbb{E}\left(\frac{1}{T}Q_i^T\right) \quad (55)$$

$$\geq \lambda_i - \epsilon, \quad (56)$$

where, (55) follows because $\{\beta_i^t\}_{t=0}^{\infty}$ are i.i.d. with mean λ_i and (56) follows from Lemma 4. Thus, we have shown that for every $\epsilon > 0$, $\exists$ a $\{p(\mathcal{S})\}_{\mathcal{S}\in\mathbb{S}}$ such that (48) is satisfied. $\square$

Proof of Lemma 2: By assumption, $\mathbf{Q}^t$ is stable. By Lemma 5 in [5], it follows that Q_i^t is substable $\forall\ i \in \mathcal{M}$. So,

$$\lim_{x\to\infty}\limsup_{t\to\infty}\ \text{Pr}\{Q_i^t > x\} = 0. \quad (57)$$

Also, note that $\forall\ x, \epsilon > 0$ we can find a t_0 such that $\forall\ t > t_0$,

$$\text{Pr}\{Q_i^t > t\epsilon\} \leq\ \text{Pr}\{Q_i^t > x\}, \quad (58)$$

and so,

$$\limsup_{t\to\infty}\ \text{Pr}\{Q_i^t > t\epsilon\} \leq \limsup_{t\to\infty}\ \text{Pr}\{Q_i^t > x\}. \quad (59)$$

Note that the left hand side of equation (59) does not depend on x and so we have

$$\limsup_{t\to\infty}\ \text{Pr}\{Q_i^t > t\epsilon\} \leq \lim_{x\to\infty}\limsup_{t\to\infty}\ \text{Pr}\{Q_i^t > x\} = 0, \quad (60)$$

which implies that

$$\lim_{t\to\infty}\ \text{Pr}\left\{\frac{Q_i^t}{t} > \epsilon\right\} = 0. \quad (61)$$

In other words,

$$\left(\frac{Q_i^t}{t}\right) \xrightarrow{P} 0, \quad (62)$$

where we use $\xrightarrow{P}$ to denote convergence in probability. Note that we need to show more than convergence in probability. Precisely, we need $\mathcal{L}^1$ convergence.

By (52) we have,

$$Q_i^t = \sum_{j=0}^{t-1}\left(\beta_i^j - R_i^j\right). \quad (63)$$

By, the strong law of large numbers,

$$\frac{1}{t}\sum_{j=0}^{t-1}\beta_i^j \xrightarrow{a.s.} \lambda_i. \tag{64}$$

Now, we use a Lemma from Billingsley [28].

Lemma 6: $X_n \xrightarrow{P} X$ iff for every subsequence X_{n_k} $\exists$ a further subsequence $X_{n_{k(i)}}$ such that $X_{n_{k(i)}} \xrightarrow{a.s.} X$ as $i \to \infty$.

By Lemma 6 and (62) for any subsequence $\left(\frac{Q_i^{t_k}}{t_k}\right)$, $\exists$ a further subsequence $\left(\frac{Q_i^{t_{k(l)}}}{t_{k(l)}}\right)$ such that

$$\left(\frac{Q_i^{t_{k(l)}}}{t_{k(l)}}\right) \xrightarrow{a.s.} 0 \text{ as } l \to \infty. \tag{65}$$

It follows from (63), (64) and Lemma 6 that,

$$\frac{1}{t}\sum_{j=0}^{t-1}R_i^j \xrightarrow{P} \lambda_i. \tag{66}$$

Now since $\frac{1}{t}\sum_{j=0}^{t-1}R_i^j$ is bounded, we can apply the Bounded Convergence Theorem and conclude that,

$$\mathbb{E}\left[\frac{1}{t}\sum_{j=0}^{t-1}R_i^j\right] \to \lambda_i. \tag{67}$$

Dividing both sides of (63) by t and taking averages gives us

$$\mathbb{E}\left(\frac{1}{t}Q_i^t\right) = \lambda_i - \mathbb{E}\left[\frac{1}{t}\sum_{j=0}^{t-1}R_i^j\right] \to 0, \tag{68}$$

as required. $\square$

Proof of Lemma 5: We have

$$\begin{aligned}
\mathbb{E}\left(\frac{1}{T}\sum_{t=0}^{T-1}R_i^t\right) &= \frac{1}{T}\sum_{t=0}^{T-1}\mathbb{E}(R_i^t) && (69)\\
&= \frac{1}{T}\sum_{t=0}^{T-1}\left[\sum_{\mathcal{S}\in\mathbb{S}}\mathbb{E}(R_i^t|S(t)=\mathcal{S})\ \Pr\{S(t)=\mathcal{S}\}\right] && (70)\\
&= \frac{1}{T}\sum_{t=0}^{T-1}\left[\sum_{\mathcal{S}\in\mathbb{S}}q_{i|\mathcal{S}}\ \Pr\{S(t)=\mathcal{S}\}\right] && (71)\\
&= \sum_{\mathcal{S}\in\mathbb{S}}q_{i|\mathcal{S}}\left[\frac{1}{T}\sum_{t=0}^{T-1}\ \Pr\{S(t)=\mathcal{S}\}\right] && (72)\\
&= \sum_{\mathcal{S}\in\mathbb{S}}q_{i|\mathcal{S}}p(\mathcal{S}). && (73)
\end{aligned}$$

$\square$

C. *Proof of Lemma 1*

We construct parallel *dominant* ALOHA systems in which one of the queues continues to transmit *dummy* packets even when the queue is empty. Dummy packets cause interference but the successful reception of a dummy packet has no significance. It has been shown [4] that for the collision channel model, these dominant systems stochastically dominate[4] the original ALOHA system in the sense that if both the dominant system and the original system have the same initial queue sizes, both systems have the same arrivals in every slot, and have the same "coin tossing" outcomes (that determine transmission attempts) in every slot, then the queue sizes in the dominant system are necessarily not smaller than those in the original system. As a result, conditions for stability of dominant systems are sufficient for stability of the original system.

Consider a two user ALOHA system in which queue 1 transmits dummy packets when it is empty. The probability of success for a packet from queue 2 is always $p_2\overline{p}_1 q_{2|\{2\}} + p_1 p_2 q_{2|\{1,2\}}$. However, in the original ALOHA system, the probability of success in queue 2 would be $p_2\overline{p}_1 q_{2|\{2\}} + p_1 p_2 q_{2|\{1,2\}}$ if queue 1 were nonempty and $p_2 q_{2|\{2\}}$ otherwise. If

$$p_2 q_{2|\{2\}} \geq p_2\overline{p}_1 q_{2|\{2\}} + p_1 p_2 q_{2|\{1,2\}} \Longleftrightarrow Q_2 \geq 0, \tag{74}$$

the probability of success in queue 2 in the "dummy" packet transmitting system would always be lower than that in the original system. Now consider queue 1. Using standard *G/G/1* queue results, it follows that the average probability of success seen in queue 1 is

$$p_1 q_{1|\{1\}}\left(1 - \frac{\lambda_2}{p_2\overline{p}_1 q_{2|\{2\}} + p_1 p_2 q_{2|\{1,2\}}}\right) + \left(p_1\overline{p}_2 q_{1|\{1\}} + p_1 p_2 q_{1|\{1,2\}}\right)\left(\frac{\lambda_2}{p_2\overline{p}_1 q_{2|\{2\}} + p_1 p_2 q_{2|\{1,2\}}}\right), \tag{75}$$

which simplifies to

$$p_1 q_{1|\{1\}} - \frac{p_1 p_2 \lambda_2 Q_1}{\lambda_2^*}. \tag{76}$$

Now, if

$$p_1 q_{1|\{1\}} \geq p_1\overline{p}_2 q_{1|\{1\}} + p_1 p_2 q_{1|\{1,2\}} \Longleftrightarrow Q_1 \geq 0, \tag{77}$$

[4]A real random variable X is said to stochastically dominate a real random variable Y if $\forall\, z \in \mathbb{R},\ \Pr\{X > z\} \geq \Pr\{Y > z\}$. We denote this dominance by $X \geq_{st} Y$.

queue 1 in the "dummy" packet system also stochastically dominates queue 1 in the original system. This is because if (77) is true, then (76) would certainly be less than the average probability of success in queue 1 in the original ALOHA system.

Now, by definition $\lambda_2^* = p_2\overline{p}_1 q_{2|\{2\}} + p_1 p_2 q_{2|\{1,2\}}$ and so by a simple application of Loynes Theorem we find that the dummy packet system is stable (with the possible exception of boundary points) if and only if

$$\lambda_1 \leq p_1 q_{1|\{1\}} - \frac{p_1 p_2 \lambda_2 Q_1}{\lambda_2^*}, \quad \lambda_2 \leq \lambda_2^*. \tag{78}$$

If in addition $Q_1 \geq 0$ and $Q_2 \geq 0$, by stochastic dominance the original ALOHA system is also stable.

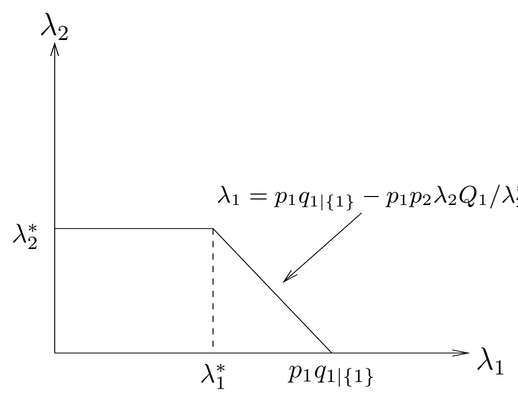


Fig. 18. Stability region of dominant system in which queue 1 transmits dummy packets.

By a parallel argument for a system in which queue 2 transmits dummy packets, we find that the original ALOHA system is stable (with the possible exception of boundary points) if $Q_1 \geq 0$ and $Q_2 \geq 0$ and

$$\lambda_2 \leq p_2 q_{2|\{2\}} - \frac{p_1 p_2 \lambda_1 Q_2}{\lambda_1^*}, \quad \lambda_1 \leq \lambda_1^*. \tag{79}$$

Combining the sufficient conditions for both the dominant systems gives the sufficiency part of the Lemma.

The necessary part of the lemma follows by an "indistinguishability" argument similar to the one used by Rao and Ephremides [4]. Consider the system in which queue 1 transmits dummy packets. If along some realizations of queue 1 of nonzero probability, queue 1 *never* empties then the original system and the dummy packet transmitting system are "indistinguishable". Thus, with a particular initial condition, if queue 1 never empties with nonzero probability in the dominant system (which is equivalent to the Markov chain being unstable) then queue 1 in the the original system must be unstable as well. Thus, for $\lambda_2 \le \lambda_2^*$, $\lambda_1 = p_1 q_{1|\{1\}} - \frac{p_1 p_2 \lambda_2 Q_1}{\lambda_2^*}$ is the boundary of the stability region of the original system as well. It follows that the conditions given in Lemma 1 are necessary for stability as well. □

D. Proof of Lemma 2

We use Lemma 1. Since we know the stability region for a fixed transmission probability vector $\mathbf{p}$, we need to find the union of all the stability regions as the parameter $\mathbf{p}$ varies over $[0,1]^2$. One way of doing this is to setup a corresponding constraint optimization problem *i.e.* for a fixed λ_1, maximize λ_2 as $\mathbf{p}$ varies over $[0,1]^2$, where λ_1 and λ_2 are related by (11) and (12). The difficulty in using known optimization techniques is that the objective function is not differentiable at a point in its domain and so the optimization has to be carried out rather explicitly.

Replacing λ_1 by x and λ_2 by y, we write the boundary given by equations (11) and (12) as

$$x = p_1 q_{1|\{1\}} - \frac{p_1 y Q_1}{(q_{2|\{2\}} - p_1 Q_2)}, \text{ for } 0 \le y < p_2(q_{2|\{2\}} - p_1 Q_2), \tag{80}$$

and

$$y = p_2 q_{2|\{2\}} - \frac{p_2 x Q_2}{(q_{1|\{1\}} - p_2 Q_1)}, \text{ for } 0 \le x < p_1(q_{1|\{1\}} - p_2 Q_1). \tag{81}$$

Note that any (x^*, y^*) which satisfies (80) does not satisfy (81) and vice versa.

Now, we consider three cases:

- *Case 1:* $Q_1 > 0$ and $Q_2 > 0$.

  Consider the following constrained maximization problem where the domain of x is unconstrained,

  $$\max_{p_2 \in [0,1]} y' = \max_{p_2 \in [0,1]} \left(p_2 q_{2|\{2\}} - \frac{p_2 x Q_2}{(q_{1|\{1\}} - p_2 Q_1)} \right), \text{ for } x \ge 0. \tag{82}$$

  Differentiating with respect to p_2,

  $$\frac{dy'}{dp_2} = q_{2|\{2\}} - \frac{x Q_2 q_{1|\{1\}}}{(q_{1|\{1\}} - p_2 Q_1)^2}. \tag{83}$$

  Equating the derivative to zero gives p_2^* as,

  $$(q_{1|\{1\}} - p_2^* Q_1) = \sqrt{\frac{x Q_2 q_{1|\{1\}}}{q_{2|\{2\}}}}. \tag{84}$$

A simple check on the second derivative shows that the maximizing p_2^* is

$$p_2^* = \frac{q_{1|\{1\}}}{Q_1} - \frac{1}{Q_1}\sqrt{\frac{xQ_2 q_{1|\{1\}}}{q_{2|\{2\}}}}. \tag{85}$$

But since p_2^* is a probability, $0 \leq p_2^* \leq 1$. Also, by (84),

$$0 \leq p_2^* \leq 1 \Longleftrightarrow \frac{q_{2|\{2\}}(q_{1|\{1\}} - Q_1)^2}{q_{1|\{1\}}Q_2} \leq x \leq \frac{q_{1|\{1\}}q_{2|\{2\}}}{Q_2}, \tag{86}$$

and so for x in the range given by (86), we have by substituting (85) in (81),

$$y'_{max} = \left(\frac{q_{1|\{1\}}q_{2|\{2\}} - \sqrt{xQ_2 q_{1|\{1\}}q_{2|\{2\}}}}{Q_1}\right)\left(1 - \sqrt{\frac{xQ_2}{q_{1|\{1\}}q_{2|\{2\}}}}\right). \tag{87}$$

Now, we consider the constraint on the domain of x as given by (81). For a fixed x, (81) is valid only for $p_2 < \frac{(q_{1|\{1\}}-x)}{Q_1}$. So, we need to show that p_2^* given by (85) is in fact less than $\frac{1}{Q_1}(q_{1|\{1\}} - x)$. Clearly,

$$p_2^* < \frac{(q_{1|\{1\}} - x)}{Q_1} \text{ iff } x < \frac{q_{1|\{1\}}Q_2}{q_{2|\{2\}}}, \tag{88}$$

and therefore, y'_{max} given by (87) is actually *really* valid only in the range of x given by the intersection of (88) and (86). Now observe that for

$$0 \leq x < \frac{q_{2|\{2\}}(q_{1|\{1\}} - Q_1)^2}{q_{1|\{1\}}Q_2}, \tag{89}$$

$$\frac{dy'}{dp_2} > 0 \ \ \forall p_2 \in [0,1].$$

This means that y' is a strictly increasing function of p_2 in the range of x given by (89). Thus, the optimizing p_2^* in this range of x is equal to one. So, we have

$$y'_{max} = q_{2|\{2\}} - \frac{xQ_2}{(q_{1|\{1\}} - Q_1)}. \tag{90}$$

But again we have to check that (81) is really valid. Note that for $x < (q_{1|\{1\}} - Q_1)$, y given by (81) is valid for any $p_2 \in [0,1]$. So, y'_{max} given by (90) is also really valid only in $x < \min\{\frac{q_{2|\{2\}}(q_{1|\{1\}}-Q_1)^2}{q_{1|\{1\}}Q_2}, (q_{1|\{1\}} - Q_1)\}$.

Figure 19 shows the behavior of y'_{max} as a function of x as given by equations (87) and (90).

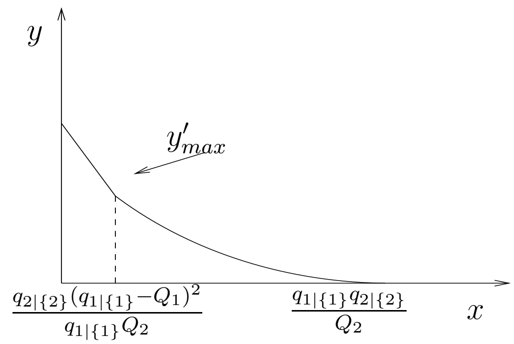


Fig. 19. The appearance of y'_{max} as a function of x.

By an exactly parallel argument applied to equation (80), we can show that

$$x'_{max} = q_{1|\{1\}} - \frac{yQ_1}{(q_{2|\{2\}} - Q_2)}, \text{ for } 0 \leq y < \frac{q_{1|\{1\}}(q_{2|\{2\}} - Q_2)^2}{q_{2|\{2\}}Q_1}, \tag{91}$$

and

$$x'_{max} = \left(\frac{q_{1|\{1\}}q_{2|\{2\}} - \sqrt{yQ_1 q_{1|\{1\}}q_{2|\{2\}}}}{Q_2} \right) \left(1 - \sqrt{\frac{yQ_1}{q_{1|\{1\}}q_{2|\{2\}}}} \right), \tag{92}$$

for

$$\frac{q_{1|\{1\}}(q_{2|\{2\}} - Q_2)^2}{q_{2|\{2\}}Q_1} \leq y \leq \frac{q_{1|\{1\}}q_{2|\{2\}}}{Q_1}.$$

The idea of the proof is this—First we assume that a rate x can be stabilized for the first user and then we find the highest rate for the second user y_{max} for which the second queue is stable. We also note from the preceding discussion that if $(q_{1|\{1\}} - Q_1) < \frac{q_{1|\{1\}}Q_2}{q_{2|\{2\}}}$, then $y_{max} = y'_{max}$ for $0 \leq x < \frac{q_{1|\{1\}}Q_2}{q_{2|\{2\}}}$ where y'_{max} is given by equations (90) for $x \in [0, \frac{q_{2|\{2\}}(q_{1|\{1\}}-Q_1)^2}{q_{1|\{1\}}Q_2})$ and (87) for $x \in [\frac{q_{2|\{2\}}(q_{1|\{1\}}-Q_1)^2}{q_{1|\{1\}}Q_2}, \frac{q_{1|\{1\}}Q_2}{q_{2|\{2\}}}]$. On the other hand if $(q_{1|\{1\}} - Q_1) \geq \frac{q_{1|\{1\}}Q_2}{q_{2|\{2\}}}$, then $y_{max} = y'_{max}$ for $0 \leq x < (q_{1|\{1\}} - Q_1)$ where y'_{max} is given by equation (90). But the problem is that the first users' rate x might, in fact, not be stabilizable. So, we interchange the users and find the best possible rate for the first user assuming the second users' rate can be stabilized. It can be checked that the point where y'_{max} ceases to be equal to y_{max} is exactly the point from where $x'_{max} = x_{max}$ and vice versa. Therefore, it is clear that the intersection of the regions below $y_{max} = f(x, q_{1|\{1\}}, q_{2|\{2\}}, Q_1, Q_2)$ and $x_{max} = f(y, q_{2|\{2\}}, q_{1|\{1\}}, Q_2, Q_1)$ where f is defined in equation (17), is actually the stability region that we seek, because for any rate pair in this intersection, there exists a $[p_1, p_2]$ such that the system is stable.

- *Case 2:* $Q_1 = 0$ and $Q_2 = 0$.

  It's obvious from equations (80) and (81), that in this case the stability region is the rectangle $0 \le x < q_{1|\{1\}}$ and $0 \le y < q_{2|\{2\}}$.

- *Case 3:* $Q_1 > 0$ and $Q_2 = 0$.

  For this case, equations (80) and (81) take the form

$$y = p_2 q_{2|\{2\}}, \text{ for } 0 \le x \le p_1(q_{1|\{1\}} - p_2 Q_1), \tag{93}$$

  and

$$x = p_1 q_{1|\{1\}} - \frac{p_1 y Q_1}{q_{2|\{2\}}}, \text{ for } 0 \le y < p_2 q_{2|\{2\}}. \tag{94}$$

  Now, if $y < q_{1|\{1\}} q_{2|\{2\}} / Q_1$ (the only case of interest), then $p_1^* = 1$ maximizes x in equation (94) and in this range

$$x_{max} = q_{1|\{1\}} - \frac{y Q_1}{q_{2|\{2\}}}. \tag{95}$$

  Also, y in equation (93) has a maximum value of $q_{2|\{2\}}$. Again, by arguments similar to *Case 1*, the stability region is the region bounded by the lines $y = q_{2|\{2\}}$ and $x = q_{1|\{1\}} - y Q_1 / q_{2|\{2\}}$.

  The case when $Q_1 = 0$ and $Q_2 > 0$ can be handled similarly.

An equivalent formulation would be to find the geometric envelope of the two parameter(p_1, p_2) family of curves which define the boundary of the stability region given in Lemma 1. □

E. *Proof of Theorem 3*

We proceed to prove equivalence between the five statements as follows.

Proof of (2) equivalent to (3)

(a) $\mathscr{S}_{\text{ALOHA}}$ is bounded by lines $\Longrightarrow$ (19): If $Q_1 = 0$ and $Q_2 = 0$, then it can be easily seen that the stability region is bounded by perpendicular lines and (19) is trivially satisfied. If either Q_1 or Q_2 is equal to zero, then by *Case 3* of the proof of Lemma 2, the stability region is bounded by lines and (19) holds. So, we consider the case $Q_1 > 0$ and $Q_2 > 0$. Under our assumptions, the point of intersection of the lines defined by (90) and (91) is $(q_{1|\{1\}} - Q_1, q_{2|\{2\}} - Q_2)$. From (90) and (91), it follows that the stability region is bounded by lines if

$$(q_{1|\{1\}} - Q_1) \leq \frac{q_{2|\{2\}}(q_{1|\{1\}} - Q_1)^2}{q_{1|\{1\}}Q_2} \tag{96}$$

and

$$(q_{2|\{2\}} - Q_2) \leq \frac{q_{1|\{1\}}(q_{2|\{2\}} - Q_2)^2}{q_{2|\{2\}}Q_1}. \tag{97}$$

which is equivalent to (19).

(b) (19) $\Longrightarrow \mathscr{S}_{\text{ALOHA}}$ is bounded by lines: For the case $Q_1 = 0$, $Q_2 = 0$ and Q_1 or Q_2 equal to zero, we know that the stability region is indeed bounded by lines. When $Q_1 > 0$ and $Q_2 > 0$, then (19) clearly implies (96) and (97) and these equations in turn imply that the stability region is bounded by lines.

Proof of (1) equivalent to (2)

(a) $\mathscr{S}_{\text{ALOHA}}$ is bounded by lines $\Longrightarrow \mathscr{S}_{\text{ALOHA}}$ is convex: Consider the line with X-intercept $q_{1|\{1\}}$ and Y-intercept $q_{2|\{2\}}$,

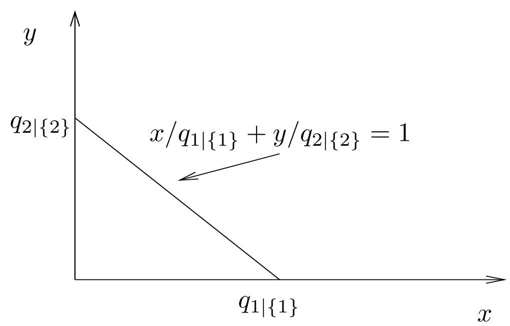


Fig. 20. The line with X-intercept $q_{1|\{1\}}$ and Y-intercept $q_{2|\{2\}}$.

$$\frac{x}{q_{1|\{1\}}} + \frac{y}{q_{2|\{2\}}} = 1, \tag{98}$$

as shown in Figure 20. By (2)$\Longleftrightarrow$(3), if the stability region is bounded by lines, then (19) holds. Also, the point of intersection of the lines is $(q_{1|\{1\}} - Q_1, q_{2|\{2\}} - Q_2)$. So, we have

$$\frac{(q_{1|\{1\}} - Q_1)}{q_{1|\{1\}}} + \frac{(q_{2|\{2\}} - Q_2)}{q_{2|\{2\}}} = 2 - \left(\frac{Q_1}{q_{1|\{1\}}} + \frac{Q_2}{q_{2|\{2\}}}\right) \geq 1, \tag{99}$$

which implies that the point $(q_{1|\{1\}} - Q_1, q_{2|\{2\}} - Q_2)$, lies above the line given by (98). Since, the stability region is bounded by lines, (99) is equivalent to the stability region being convex.

(b) $\mathscr{S}_{\text{ALOHA}}$ is convex $\Longrightarrow$ $\mathscr{S}_{\text{ALOHA}}$ is bounded by lines: We will prove that the contrapositive is true. So, assume that the stability region is not bounded by lines. Then, by proposition 2 (proved below) the stability region has a form where the boundary is a strictly convex function in the middle with or without lines at the the ends. In any case, it follows that the stability region is not convex.

Proof of (3) equivalent to (5)

We use Lemma 1. Substitute $\mathbf{p} = [1, 1]$ in Lemma 1. Then, $\lambda_1^* = q_{1|\{1,2\}}$ and $\lambda_2^* = q_{2|\{1,2\}}$ and it is easily seen that the stability region with $\mathbf{p} = [1, 1]$ coincides with $\mathscr{S}_{\text{ALOHA}}$ if and only if $\frac{q_{1|\{1,2\}}}{q_{1|\{1\}}} + \frac{q_{2|\{1,2\}}}{q_{2|\{2\}}} \geq 1$. $\square$

Proof of (3) equivalent to (4)

It can be easily verifed that when equation (19) holds, $\mathscr{S}_{\text{ALOHA}} = \mathscr{C}$. On the other hand, if equation (19) does not hold, then $\mathscr{C}$ is a triangle with vertices $(0, 0)$, $(q_{1|\{1\}}, 0)$, and $(0, q_{2|\{2\}})$ whereas from Lemma 2, $\mathscr{S}_{\text{ALOHA}}$ lies strictly inside this triangle. $\square$

Next, we claim the following.

Proposition 2: If $\mathscr{S}_{\text{ALOHA}}$ is not bounded by lines, then the strictly convex parts of f which bound the regions $\mathcal{R}_1$ and $\mathcal{R}_2$ in Lemma 2 coincide on the boundary of the stability region.

Proof:

Consider the equations,

$$x = \left(\frac{q_{1|\{1\}}q_{2|\{2\}} - \sqrt{yQ_1 q_{1|\{1\}}q_{2|\{2\}}}}{Q_2}\right)\left(1 - \sqrt{\frac{yQ_1}{q_{1|\{1\}}q_{2|\{2\}}}}\right) \tag{100}$$

and

$$y = \left(\frac{q_{1|\{1\}}q_{2|\{2\}} - \sqrt{xQ_2 q_{1|\{1\}}q_{2|\{2\}}}}{Q_1}\right)\left(1 - \sqrt{\frac{xQ_2}{q_{1|\{1\}}q_{2|\{2\}}}}\right), \tag{101}$$

which are the strictly convex parts of f characterizing regions $\mathcal{R}_2$ and $\mathcal{R}_1$ respectively. If we make the substitution $x' = \sqrt{xQ_2 q_{1|\{1\}}q_{2|\{2\}}}$ and $y' = \sqrt{yQ_1 q_{1|\{1\}}q_{2|\{2\}}}$, then (100) and (101) become

$$(q_{1|\{1\}}q_{2|\{2\}} - y')^2 = x'^2 \text{ and } (q_{1|\{1\}}q_{2|\{2\}} - x')^2 = y'^2. \tag{102}$$

Solving the equations in (102) simultaneously gives,

$$x' + y' = q_{1|\{1\}}q_{2|\{2\}}. \tag{103}$$

It follows that any (x^*, y^*) which satisfies (103) satisfies (102) as well. In other words, all the points (x, y) which satisfy (100) also satisfy (101) and vice versa and we are done.

The above claim guarantees that the non-linear portions of the function f in Lemma 2 coincide for regions $\mathcal{R}_1$ and $\mathcal{R}_2$. This implies that the non-linear boundary of $\mathscr{S}_{\text{ALOHA}}$ is a strictly convex function since f is strictly convex. $\square$

F. Proof of Theorem 4

From [29], it can be shown (as a special case) that for a finite ($N \geq 2$) user system, with symmetric arrival rate, a symmetric MPR reception model as given by (20) and transmission probability p, under the following stochastic dominance condition

$$C_1 \geq \frac{C_2}{2} \geq \cdots \geq \frac{C_i}{i} \geq \cdots \geq \frac{C_N}{N}, \tag{104}$$

the maximum stable throughput of ALOHA is given by

$$\rho_{\text{ALOHA}}(p) = \sum_{i=1}^{N} \binom{N}{i} p^i (1-p)^{N-i} C_i, \tag{105}$$

Now, note that

$$\sum_{i=1}^{N} \binom{N}{i} p^i (1-p)^{N-i} C_i \leq \max\{C_1, \cdots, C_N\}. \tag{106}$$

Clearly, if $C_N = \max\{C_1, \cdots, C_N\}$ then the above inequality is satisfied with equality for $p = 1$. By Corollary 1, it follows that $\rho_{\text{ALOHA}}(1) = \rho_{\text{ALOHA}} = \rho = \max\{C_1, \cdots, C_N\}$. The converse follows directly. $\square$

G. Proof of Theorem 5

The proof is similar to the one provided by Szpankowski [5] with appropriate modifications. We will prove the claim with induction on $N = |\mathcal{M}|$. It can be easily shown that the claim is true for $N = 2$. In fact, the region given by Theorem 5 is actually the stability region for $N = 2$ for a fixed $[p_1, p_2]$.

Now, we apply induction on N. Assume that the claim is true for ALOHA systems with number of users less than N. Now, consider the ALOHA system with N users and with reception probabilities $\mathbf{q}^{\mathcal{M}}$. Under conditions of stochastic dominance, we know that $\Theta^{\mathcal{P}}$ dominates the original ALOHA system. In system $\Theta^{\mathcal{P}}$, queues in $\mathcal{V}$ are "decoupled" from those in $\mathcal{U}$ or in other words queues in $\mathcal{V}$ appear to be a smaller version of the ALOHA system with modified reception probabilities $\mathbf{q}^{\mathcal{V}}$ as given by (27). But since $|\mathcal{V}| < N$, by the induction hypothesis if $\boldsymbol{\lambda}_{\mathcal{V}} \in \mathcal{R}(\mathbf{q}^{\mathcal{V}})$ then the queues in $\mathcal{V}$ are stable. Thus in the dominant system $\Theta^{\mathcal{P}}$, queues in $\mathcal{V}$ are stable if $\boldsymbol{\lambda}_{\mathcal{V}} \in \mathcal{R}(\mathbf{q}^{\mathcal{V}})$.

Now, we focus our attention on the set of persistent queues $\mathcal{U}$ in $\Theta^{\mathcal{P}}$. Here, we use Loynes Theorem for stability in queueing systems [17]. In order to use Loynes Theorem, it is necessary to show that the departure process Y_j^t in $\Theta^{\mathcal{P}}$ is stationary and ergodic. Let $Y_j^t(\Theta^{\mathcal{P}})$ denote the departure process of a queue j in $\Theta^{\mathcal{P}}$. Then, we have

$$Y_j^t(\Theta^{\mathcal{P}}) = R_j^t \sum_{\mathcal{S}\subseteq\mathcal{M}\backslash\{j\}} \mathbf{1}\left[\mathcal{S} = \left\{\{k : R_k^t \mathbf{1}[\overline{Q}_k^t > 0] = 1, k \in \mathcal{V}\} \cup \{k : R_k^t = 1, k \in \mathcal{U}\}\right\}\right] \mathbf{1}[Z_j^t(\mathcal{S}) = 1]. \tag{107}$$

From the above equation it follows that if the Markov chain $\overline{\mathbf{Q}}_{\mathcal{V}}^t$ is stationary and ergodic, then $Y_j^t(\Theta^{\mathcal{P}})$ for $j \in \mathcal{U}$ is also stationary and ergodic, since the coin tosses (determining transmission attempts) and packet reception events are independent from slot to slot. But, since $\boldsymbol{\lambda}_{\mathcal{V}} \in \mathcal{R}(\mathbf{q}^{\mathcal{V}})$, we know that $\overline{\mathbf{Q}}_{\mathcal{V}}^t$ is stable and also ergodic. So, if we initialize $\overline{\mathbf{Q}}_{\mathcal{V}}^t$ with its unique stationary distribution, then the departure process $Y_j^t(\Theta^{\mathcal{P}})$ for persistent queue j is also stationary and ergodic in $\Theta^{\mathcal{P}}$. Now, we can apply Loynes theorem to this persistent queue in $\Theta^{\mathcal{P}}$. So, if $\lambda_j < \mathbb{E}(Y_j^t(\Theta^{\mathcal{P}}))$, queue j is stable in $\Theta^{\mathcal{P}}$. In the above, the expectation is with respect to the stationary distribution of $\overline{\mathbf{Q}}_{\mathcal{V}}^t$. It can be shown that $\mathbb{E}(Y_j^t(\Theta^{\mathcal{P}})) = P_{\Theta^{\mathcal{P}}}^j$ where $P_{\Theta^{\mathcal{P}}}^j$ is given by (28).

Thus, if $\boldsymbol{\lambda}_{\mathcal{V}} \in \mathcal{R}(\mathbf{q}^{\mathcal{V}})$ and $\lambda_j < P_{\Theta^{\mathcal{P}}}^j$ for all $j \in \mathcal{U}$, each queue in $\Theta^{\mathcal{P}}$ is stable. Now, we use the fact that for a countable state space irreducible and aperiodic Markov chain, stability of individual queues implies stability of the Markov Chain [5]. Thus, if $\lambda_{\mathcal{V}} \in \mathcal{R}(\mathbf{q}^{\mathcal{V}})$ and $\lambda_j < P_{\Theta^{\mathcal{P}}}^j$ for all $j \in \mathcal{U}$, then $\Theta^{\mathcal{P}}$ is stable and by virtue of the stochastic dominance, the original ALOHA system is also stable. Clearly, the partition $\mathcal{P}$ can be arbitrary with $|\mathcal{V}| < N$; hence the union over all the partitions. □

H. Computation of $\mathbf{q}$ *with linear front-ends*

Let v_i denote the ith column of $\mathbf{V}$ *i.e.*, $\mathbf{V} = [v_1, v_2]$. Also, let f_i denote the ith row of F. When user $i, (i = 1, 2)$ transmits alone, then it is easy to see that

$$\begin{aligned} q_{i|i} &= \Pr\left\{\frac{|r_{ii}|^2}{\mathbb{E}(|w_i|^2)} > \beta\right\}, \\ &= \Pr\left\{\frac{|h_i|^2|f_i v_i|^2}{f_i f_i^H} > \beta\right\}, \\ &= \exp\left(-\frac{\beta f_i f_i^H}{|f_i v_i|^2\sigma_{h_i}^2}\right), \end{aligned}$$

since we assume that the fading is Rayleigh.

Now, without loss of generality we compute the probability of success of the first user when both users transmit in a slot. In this case,

$$\begin{aligned} q_{1|\{1,2\}} &= \Pr\left\{\frac{|r_{11}|^2}{|r_{12}|^2 + \mathbb{E}(|w_1|^2)} > \beta\right\}, \\ &= \Pr\left\{\frac{|h_1|^2|f_1 v_1|^2}{|h_2|^2|f_1 v_2|^2 + f_1 f_1^H} > \beta\right\}. \end{aligned}$$

Since $|h_1|^2 \sim exp(1/\sigma_{h_1}^2)$, $|h_2|^2 \sim exp(1/\sigma_{h_2}^2)$ and they are independent, we can evaluate the above probability using the joint density of $|h_1|^2$ and $|h_2|^2$. Thus, we have

$$q_{1|\{1,2\}} = \frac{1}{\sigma_{h_1}^2\sigma_{h_2}^2}\int_{\frac{\beta f_1 f_1^H}{|f_1 v_1|^2}}^{\infty} \exp(-\frac{x}{\sigma_{h_1}^2})\left(\int_0^{\frac{x|f_1 v_1|^2 - \beta f_1 f_1^H}{\beta |f_1 v_2|^2}} \exp(-\frac{y}{\sigma_{h_2}^2})\, dy\right)\, dx.$$

After a few algebraic manipulations we get,

$$q_{1|\{1,2\}} = \exp\left(-\frac{\beta f_1 f_1^H}{|f_1 v_1|^2\sigma_{h_1}^2}\right)\left[\frac{\sigma_{h_1}^2|f_1 v_1|^2}{\sigma_{h_1}^2|f_1 v_1|^2 + \beta|f_1 v_2|^2}\right].$$

A similar calculation can be done for the second user as well. Note that under our model with linear front-ends, $q_{i|i} - q_{i|\{1,2\}} = Q_i \geq 0$ and thus our stability results apply to this $\mathbf{q}$.

I. Proof of Theorem 6

Let Q_i^t denote the queue length of the ith user in time slot t. Let $\beta_i^t \overset{i.i.d}{\sim}$ Bernoulli(r), denote the number of packets that arrive at the ith users' queue in time slot t. Let $F(x, y)$ be the moment generating function of the joint arrival process. Thus, $\forall |x| \leq 1, |y| \leq 1, t \in \mathbb{N}$

$$F(x, y) = \mathbb{E}\left(x^{\beta_1^t} y^{\beta_2^t}\right) = (xr + \overline{r})(yr + \overline{r}). \tag{108}$$

Then, from the queue evolution equation (5), it can be seen that

$$\begin{aligned}\mathbb{E}\left(x^{Q_1^{(t+1)}}y^{Q_2^{(t+1)}}\right) &= F(x,y)\Bigg[\mathbb{E}\left(\mathbf{1}[Q_1^t=0,Q_2^t=0]\right)\\ &+\left(\frac{pa}{x}+p(1-a)+(1-p)\right)\mathbb{E}\left(x^{Q_1^t}\mathbf{1}[Q_1^t>0,Q_2^t=0]\right)\\ &+\left(\frac{pa}{y}+p(1-a)+(1-p)\right)\mathbb{E}\left(y^{Q_2^t}\mathbf{1}[Q_1^t=0,Q_2^t>0]\right)\\ &+\left(\left(pa+p^2(b-a)\right)\left(\frac{1}{x}+\frac{1}{y}\right)+(1-p)^2+2p(1-p)(1-a)+p^2(1-2b)\right)\\ &\times\mathbb{E}\left(x^{Q_1^t}y^{Q_2^t}\mathbf{1}[Q_1^t>0,Q_2^t>0]\right)\Bigg]. \qquad (109)\end{aligned}$$

By Lemma 1, if $r<(pa+p^2(b-a))$, then the ALOHA system is stable. Since, (Q_1^t,Q_2^t) is an irreducible, aperiodic Markov chain, stability is equivalent to existence of a unique stationary (limiting) distribution. Let $G(x,y)$ be the moment generating function of the joint stationary queue process *viz.*,

$$G(x,y)=\lim_{t\to\infty}\mathbb{E}\left(x^{Q_1^t}y^{Q_2^t}\right). \qquad (110)$$

Now, note that

$$\begin{aligned}G(0,0) &= \lim_{t\to\infty}\mathbb{E}\left(\mathbf{1}[Q_1^t=0,Q_2^t=0]\right), && (111)\\ G(x,0)-G(0,0) &= \lim_{t\to\infty}\mathbb{E}\left(x^{Q_1^t}\mathbf{1}[Q_1^t>0,Q_2^t=0]\right), && (112)\\ G(0,y)-G(0,0) &= \lim_{t\to\infty}\mathbb{E}\left(y^{Q_2^t}\mathbf{1}[Q_1^t=0,Q_2^t>0]\right), && (113)\\ G(x,y)+G(0,0)-G(x,0)-G(0,y) &= \lim_{t\to\infty}\mathbb{E}\left(x^{Q_1^t}y^{Q_2^t}\mathbf{1}[Q_1^t>0,Q_2^t>0]\right). && (114)\end{aligned}$$

Using an approach similar to ([21], [22]), from (109) it follows that $G(x,y)$ satisfies the following functional equation.

$$\begin{aligned}G(x,y) &= F(x,y)\Big(C(x,y)(G(x,y)+G(0,0)-G(x,0)-G(0,y)) \qquad (115)\\ &+A(x,y)(G(x,0)-G(0,0))+B(x,y)(G(0,y)-G(0,0))+G(0,0)\Big),\end{aligned}$$

where

$$\begin{aligned} A(x,y) &= \frac{pa}{x} + p(1-a) + (1-p) \\ B(x,y) &= \frac{pa}{y} + p(1-a) + (1-p) \end{aligned}$$

$$C(x,y) = (p^2 b + p\overline{p}a)\left(\frac{1}{x} + \frac{1}{y}\right) + \overline{p}^2 + 2p\overline{p}(1-a) + p^2(1-2b). \tag{116}$$

As observed by Sidi [21], the above functional equation cannot be solved for $G(x,y)$ easily. However, (115) can be used to compute the average delay as follows: First, use the fact that $G(1,1) = 1$ and $G(1,0) = G(0,1)$(symmetry) to find

$$G(1,0)(2p^2(b-a) + pa) - G(0,0)p^2(b-a) = pa + p^2(b-a) - r. \tag{117}$$

Let $G_1(x,y) \triangleq dG(x,y)/dx$. From (115), we can show that

$$G_1(1,1) = \frac{p^2(b-a)G_1(1,0) + r\overline{r}}{pa + p^2(b-a) - r} \tag{118}$$

and

$$\left.\frac{dG(x,x)}{dx}\right|_{x=1} = 2r + \frac{(pa + 2p^2(b-a))G_1(1,0)}{pa + p^2(b-a) - r} + \frac{(r^2 + 2r - 4r(p^2(b-a) + pa))}{2(pa + p^2(b-a) - r)}. \tag{119}$$

It follows by symmetry that

$$\left.\frac{dG(x,x)}{dx}\right|_{x=1} = 2G_1(1,1). \tag{120}$$

Substituting (120) in (119) and eliminating $G_1(1,0)$ from (118) and (119) gives

$$G_1(1,1) = \frac{1}{pa}\left[r + \frac{(2r^2 - 2r^2 pa - r^2 p^2(b-a))}{2(pa + p^2(b-a) - r)}\right]. \tag{121}$$

Since $G_1(1,1)$ is equal to the mean queue length of the users, a simple application of Little's theorem gives us the mean queueing delay (D) as

$$
\begin{aligned}
D &= \frac{G_1(1,1)}{r} \\
&= \frac{1}{pa}\left[1+\frac{(2r-2rpa-rp^2(b-a))}{2(pa+p^2(b-a)-r)}\right] \\
&= \frac{1}{a}\left[\frac{a(1-r)+p(b-a)(1-r/2)}{pa+p^2(b-a)-r}\right]. \qquad (122)
\end{aligned}
$$

$\square$.

J. Proof of Lemma 3

The difficulty in finding the optimizing transmission probability (p^*) is that it is the solution to a constrained optimization problem and the objective function is not convex. Mathematically, the problem can be cast as follows.

$$
p^* = \underset{\substack{pa+p^2(b-a)-r>0,\\ p\in[0,1]}}{\arg\min} \frac{1}{a}\left[\frac{a(1-r)+p(b-a)(1-r/2)}{pa+p^2(b-a)-r}\right]. \qquad (123)
$$

We first focus on a looser constrained optimization problem. So, let

$$
p' = \underset{pa+p^2(b-a)-r>0}{\arg\min} \frac{1}{a}\left[\frac{a(1-r)+p(b-a)(1-r/2)}{pa+p^2(b-a)-r}\right]. \qquad (124)
$$

The roots of the quadratic equation $p^2(b-a)+pa-r=0$ are

$$
\begin{aligned}
s_1 &= \frac{a-\sqrt{a^2-4(a-b)r}}{2(a-b)} \\
s_2 &= \frac{a+\sqrt{a^2-4(a-b)r}}{2(a-b)}.
\end{aligned}
$$

It is easy to check that s_1 and s_2 are real. Thus, the constraint $p^2(b-a)+pa-r>0$ is equivalent to $s_1<p<s_2$. Next, we differentiate the objective function in (124) and equate it to zero. That yields

$$
\left[-p^2(a-b)^2(1-r/2)+2a(a-b)p(1-r)-a^2(1-r)+r(a-b)(1-r/2)\right]=0. \qquad (125)
$$

The roots of the above quadratic equation are

$$p_1 = \frac{a(1-r) - \sqrt{0.5r}\sqrt{2(a-b)(1-r/2)^2 - a^2(1-r)}}{(a-b)(1-r/2)} \quad (126)$$

$$p_2 = \frac{a(1-r) + \sqrt{0.5r}\sqrt{2(a-b)(1-r/2)^2 - a^2(1-r)}}{(a-b)(1-r/2)}.$$

Now, we have to consider two cases:

Case 1: $a^2 - 2(a-b) \leq 0$.

It can be easily shown that in this case p_1 and p_2 are real. Equivalently for this case, $0 \leq b \leq a - a^2/2$. Note that for $a \leq 1$, $a/2 \leq a - a^2/2$.

Next, algebraic manipulations yield, (i) $s_1 < p_1 < s_2 < p_2$ and (ii) $0 \leq s_1 \leq 1$. Note that (i) implies that the objective function in (124) is strictly decreasing on (s_1, p_1) and is strictly increasing on (p_1, s_2). This implies that p_1 is a local minima. The constraint $p \in (s_1, s_2)$ and the fact that $s_2 < p_2$ together imply that $p' = p_1$.

Next, we turn our attention to the actual constrained optimization problem in (123). Here, (i) and (ii) imply that $p' > 0$. So, we need to check when $p' \leq 1$. If $p' \leq 1$, then obviously $p^* = p'$. Else if $p' > 1$, (i) and (ii) together imply that $p^* = 1$. This is because the objective function is strictly decreasing on (s_1, p') and so the optimal p lies on the boundary *viz.*, 1. Thus, we can conclude that,

$$\begin{aligned} p^* &= \min(1, p') \\ &= \min\left(1, \frac{a(1-r) - \sqrt{0.5r}\sqrt{2(a-b)(1-r/2)^2 - a^2(1-r)}}{(a-b)(1-r/2)}\right). \end{aligned} \quad (127)$$

Again after some manipulations it can be shown that (127) is equivalent to

$$p^* = \begin{cases} 1, & r \in [0, r^*] \\ \frac{a(1-r) - \sqrt{0.5r}\sqrt{2(a-b)(1-r/2)^2 - a^2(1-r)}}{(a-b)(1-r/2)}, & r \in (r^*, r_{max}), \end{cases} \quad (128)$$

where

$$r^* = \begin{cases} 1 - \left(\frac{a+b}{2} - \frac{ab}{a-b}\right) - \sqrt{1 - (a-b) + \left(\frac{a+b}{2} - \frac{ab}{a-b}\right)^2}, & 0 \leq b < a/2 \\ b, & a/2 \leq b \leq a - a^2/2 \end{cases} \quad (129)$$

and

$$r_{max} = \begin{cases} \frac{a^2}{4(a-b)}, & 0 \le b < a/2 \\ b, & a/2 \le b \le a - a^2/2. \end{cases} \tag{130}$$

Case 2: $a^2 - 2(a-b) > 0$.

This case is equivalent to $a - a^2/2 < b \le \min\{0.5, a\}$. This case is much easier to handle. It follows that in this case, the derivative of the objective function with respect to p in (124) is always negative which implies that the objective function is a strictly decreasing function of p. In this case, it can also be easily shown that $s_1 < 1 < s_2$ where s_1 and s_2 as defined before are the roots of $p^2(b-a) + pa - r = 0$. The additional constraint that $p \in [0, 1]$ together with $p \in (s_1, s_2)$ implies that the optimal transmission probability in this case is always one.

Combining the results from *Cases 1* and *2*, we get

$$p^* = \begin{cases} 1, & r \in [0, r^*] \\ \frac{a(1-r) - \sqrt{0.5r}\sqrt{2(a-b)(1-r/2)^2 - a^2(1-r)}}{(a-b)(1-r/2)}, & r \in (r^*, r_{max}), \end{cases} \tag{131}$$

where

$$r^* = \begin{cases} 1 - \left(\frac{a+b}{2} - \frac{ab}{a-b}\right) - \sqrt{1 - (a-b) + \left(\frac{a+b}{2} - \frac{ab}{a-b}\right)^2}, & 0 \le b < a/2 \\ b, & a/2 \le b \le \min\{0.5, a\} \end{cases} \tag{132}$$

and

$$r_{max} = \begin{cases} \frac{a^2}{4(a-b)}, & 0 \le b < a/2 \\ b, & a/2 \le b \le \min\{0.5, a\}. \end{cases} \tag{133}$$

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