Delay Analysis of Slotted ALOHA in Capture Channels for the Two User Case

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Abstract

We consider the effect of capture on delay in buffered slotted ALOHA based random access systems for the two user case. In a capture channel, at most one user can have a successful packet transmission when both users transmit. For symmetric Bernoulli arrival rates and a symmetric capture model, we characterize the average delay in terms of the retransmission probability and the capture model parameters. Further, we provide exact expressions for the delay minimizing retransmission probability for all possible capture models. We find that as soon as there is a non-zero capture probability, there is a non-empty subset of ALOHA stabilizable rates for which the optimal retransmission probability is one. We also find that when the capture probability crosses a threshold, the optimal retransmission probability is one for all stabilizable rates.

1 Introduction

We consider the problem of characterizing the average delay in buffered slotted ALOHA based random access systems with capture. By delay we mean the number of slots elapsed from the moment a packet is generated until it is successfully received. Delay is a primary performance metric in wireless networks since it determines the kind of applications that the network can support. Voice and other multimedia traffic are very sensitive to delay experienced by packets; hence the need to analyze delay performance in wireless networks.

Delay incurred by packets is determined by the joint performance of PHY and MAC layers. In this work, we focus on the delay performance of the slotted ALOHA protocol for a PHY layer that enables successful packet reception even when more than one user transmits in a slot (referred to as "capture"). We assume that every user in the multiple access system has unlimited buffer capacity to store newly generated and backlogged packets. This captures the queueing aspect of this random access problem. Sidi and Segall [1] first looked at the problem of analyzing delay in ALOHA type systems. They found the exact average delay in a two user system with symmetric arrival rates and

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retransmission probabilities. They also found the optimal retransmission probabilities to minimize delay. However, they had assumed a collision channel model for packet success which fails to capture effects of fading in wireless channels. Further, Nain [2] calculated the exact delay in the two user case for asymmetric arrivals and retransmission probabilities assuming a collision channel. In both these works, the technique used to find delay involved a functional equation in the generating function of the joint stationary queue length distribution. This functional equation can be solved by formulating a very complex Riemann-Hilbert boundary value problem [3, 4]. It is indeed quite surprising that there are no results on the "exact" delay of ALOHA for the finite user, infinite buffer queueing model apart from these two. Takagi and Kleinrock [5] used a similar approach to find average delay in a two-user buffered CSMA/CD system with a collision channel. There is also a line of work which computes bounds on average delay for $N > 2$ for the collision channel ([6, 7, 8, 9]) and for a more general symmetric Multipacket Reception (MPR) model [10]. There are quite a few other results on delay of ALOHA but they are for different queueing models *viz.,* the infinite user single buffer model and the finite user single buffer model. These models do not quite capture the interdependence amongst the queues and its effect on delay. The limited results found suggest that characterizing delay in buffered ALOHA systems is a nontrivial task.

As mentioned earlier, most of the prior work assumed a collision channel for packet success. This was motivated by the underlying wireline channel medium where only one user could be accommodated at a time in the available bandwidth. However, this is far from true in wireless channels. On the one hand even if a single user transmits in a slot it is possible for his transmission to be unsuccessful because of fading. On the other hand, it is possible that if many users transmit in the same slot the user with the strongest received power gets "captured"(see [11, 12] and the references therein). In this work, we allow the packet reception process to be probabilistic in order to model capture and other fading effects that are unique to wireless channels.

Our main findings are that as soon as there is a nonzero capture probability there is a non-empty subset of stabilizable rates for which the delay minimizing retransmission probability is one. We also observe a threshold effect; when the capture capability crosses a threshold, the delay minimizing retransmission probability is one *for all stable rates.* In fact, retransmission probability one also stabilizes all stable rates beyond the same threshold.

The rest of this paper is organized as follows. In Section 2, we provide the system model and parametrize the capture channel. In Section 3, we provide exact expressions for the delay and the optimal retransmission probabilities for minimizing delay. In Section 4, we provide some analytical insights into the results of Section 3. Finally, we conclude in Section 5.

2 System Model

The system consists of two users, each having an infinite buffer for storing arriving and backlogged packets, communicating with a common receiver. The channel is slotted in time and a slot duration equals the packet transmission time. Packets are assumed to be of equal length for the users. The arrivals to the ith user are i.i.d. Bernoulli (r) in every slot. The arrivals are independent across users. Let p be the retransmission probability of both users. We model the phenomenon of capture with the following symmetric packet reception model. Let a denote the probability of success of either user when that user is transmitting alone. Let b denote the probability of success of the first user alone when both users transmit simultaneously. Then, b is also the probability of success of the second user alone when both of them transmit together. Capture by definition implies that both users cannot have simultaneous successful packet receptions. Clearly $a \leq 1$ and $b \leq 0.5$. In addition, we assume that $b \leq a$.

3 Delay and Optimal Retransmission Probability

Theorem 1 *Let D be the average delay for either user in the symmetric capture channel. If $r < pa + p^2(b-a)$,*

$$D = \frac{1}{a}\left[\frac{a(1-r) + p(b-a)(1-r/2)}{pa + p^2(b-a) - r}\right]. \tag{1}$$

Proof: We present a sketch of the proof. It has been shown that if $r < pa + p^2(b-a)$, then the queues in the ALOHA system have a proper limiting (stationary) distribution (*viz.,* the system is stable) [13]. Next, we show that the generating function of the joint stationary distribution of the two queues satisfies a functional equation. Even though this functional equation is difficult to solve explicitly, the mean number of packets in the queues can be found by evaluating the derivative of the generating function. Finally, a simple application of Little's theorem gives the desired result. For details refer to [14]. □

From (1), we observe that the delay is a decreasing function of r as expected.

Next, we look at the problem of optimizing the retransmission probability (p^*) to minimize the average delay. We find that as soon as there is capture capability, the optimal retransmission probability is one for a set of arrival rates of the form $[0, r^*]$ with $r^* > 0$. We call this degenerate instance of ALOHA with retransmission probability one "Dumb Scheduling" since it is equivalent to the following simple MAC—transmit if you have packets [15]. Thus, dumb scheduling is delay optimal in the class of ALOHA protocols with fixed retransmission probability for small arrival rates in capture channels.

Lemma 1 *Let p^* be the optimal retransmission probability for minimizing delay in the capture channel. Then,*

$$p^* = \begin{cases} 1 & r \in [0, r^*] \\ \frac{a(1-r)-\sqrt{0.5r}\sqrt{2(a-b)(1-r/2)^2 - a^2(1-r)}}{(a-b)(1-r/2)} & r \in (r^*, r_{max}), \end{cases} \tag{2}$$

where,

$$r^* = \begin{cases} 1 - \left(\frac{a+b}{2} - \frac{ab}{a-b}\right) - \sqrt{1-(a-b) + \left(\frac{a+b}{2} - \frac{ab}{a-b}\right)^2} & 0 \leq b < a/2 \\ b & a/2 \leq b \leq \min\{0.5, a\}, \end{cases} \tag{3}$$

and

$$r_{max} = \begin{cases} \frac{a^2}{4(a-b)} & 0 \leq b < a/2 \\ b & a/2 \leq b \leq \min\{0.5, a\}. \end{cases} \tag{4}$$

Proof: The proof involves solving a constraint optimization problem. Since (1) gives us the delay for a fixed p and capture model, we have to optimize (1) with respect to p. The constraints on p are that $p \in [0,1]$ (since it is a probability) and p should stabilize the ALOHA system. Thus, the problem can be cast as

$$p^* = \underset{\substack{pa+p^2(b-a)-r>0 \\ p\in[0,1]}}{\arg\min} \; \frac{1}{a}\left[\frac{a(1-r)+p(b-a)(1-r/2)}{pa+p^2(b-a)-r}\right] \tag{5}$$

Since the objective function in this case is not convex, the optimization has to be carried out rather explicitly. Refer to [14] for details. □

Lemma 1 gives p^* explicitly in terms of the capture channel parameters and the arrival rate r. As a direct consequence of Lemma 1 we have the following theorem.

Theorem 2 *For the capture channel with $a > 0, b > 0$, the optimal retransmission probabilities can take only two possible forms viz.,*

1. *If $b < a/2$, then the optimal retransmission probability is one for a non-empty proper subset of all stable rates of the form $[0, r^*]$ with $0 < r^* < r_{max}$.*
2. *If $b \geq a/2$, then the optimal retransmission probability is one for all stable arrival rates.*

Proof: For a fixed $a > 0$, from Lemma 1 note that,

$$r^*|_{b=0} = 0.$$

It can be shown that r^* is a strictly increasing function of b for a fixed value of $a > 0$. Thus, as soon as we have capture ($b > 0$), $r^* > 0$ and there is a set of rates $[0, r^*]$ for which $p^* = 1$ is the best policy for minimizing delay. As long as $b < a/2$, we have $r^* < r_{max}$. On the other hand when $b \in [a/2, \min\{0.5, a\})$, from Lemma 1

$$(r^* = r_{max})|_{b\in[a/2,\min\{0.5,a\})} = b, \tag{6}$$

and so $p^* = 1$ is delay optimal for any rate which is stabilizable. □

Note that for $b < a/2$, $r^* < r_{max}$, and there is a set of rates for which the optimal retransmission probability p^* is still a function of the arrival rate (r). Thus, $b = a/2$ also happens to be the point where the optimal retransmission probability ceases to be a function of the arrival rate. We refer to r^* as the *critical* rate since rates below r^* are delay optimized by dumb scheduling. In [13], it has already been shown that r_{max} is the maximum stable arrival rate for the capture model.

Figure 1 shows the generic optimal retransmission probabilities as a function of the capture channel parameters. It is interesting to compare the structure of the stability region of ALOHA (see [13]) along the equal rate line with the optimal retransmission probability for different capture models. Note that $b = a/2$ is also the point from which dumb scheduling is optimal from a stability viewpoint. Thus, dumb scheduling is optimal from both delay and stability viewpoints when $b \geq a/2$.

Figure 2 shows the set of retransmission probabilities that stabilize the ALOHA system for different arrival rates in a weak capture ($b < a/2$) case. The maximum and minimum stabilizing retransmission probabilities are the solution to the equation

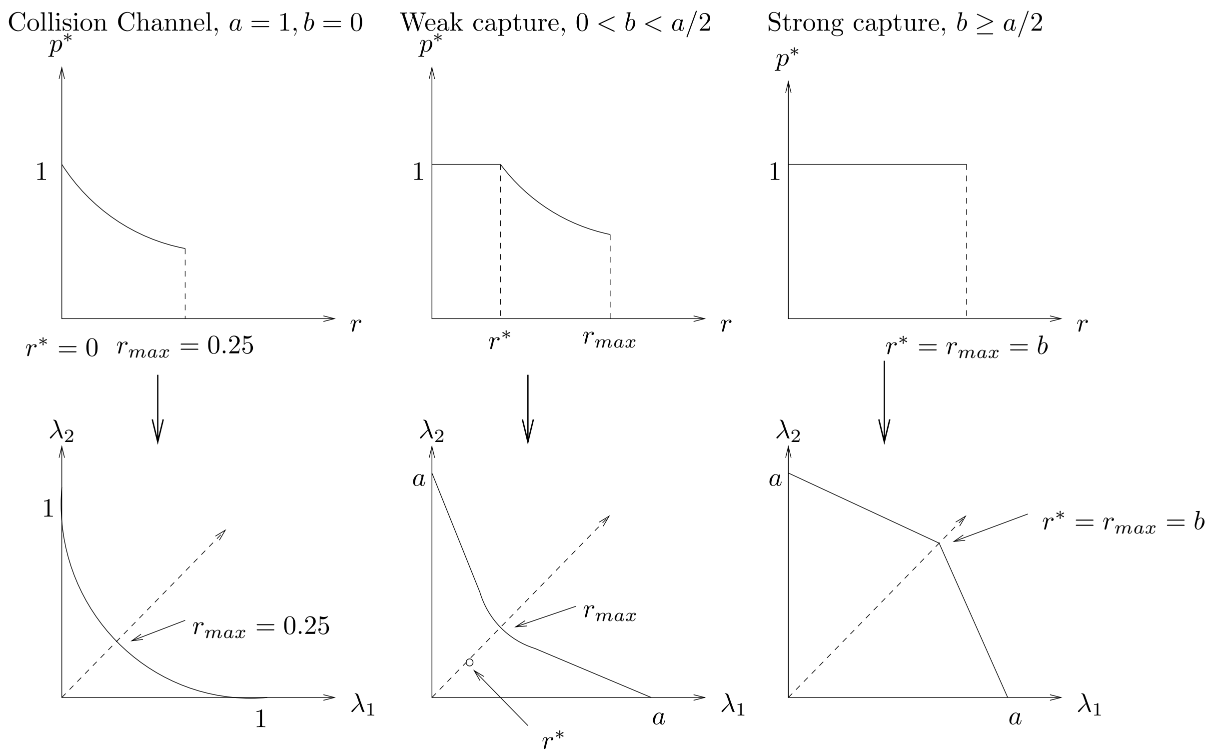

Figure 1: Generic optimal retransmission probabilities for capture channels.

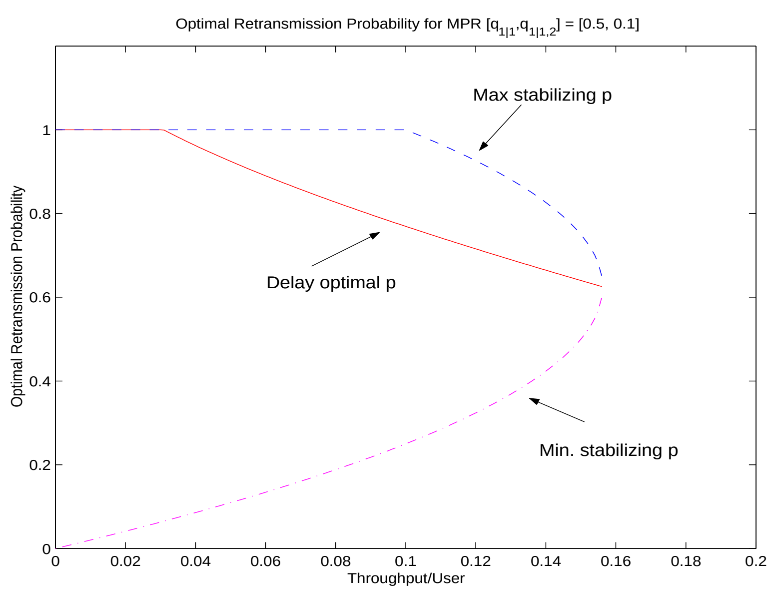

Figure 2: The optimal and stabilizing retransmission probabilities for $a = 0.5$, $b = 0.1$

$p^2(b - a) + pa - r = 0$ and thus form a parabola which is truncated since the maximum retransmission probability can be at most one. The point at which the maximum and minimum retransmission probabilities coincide corresponds to the maximum stable arrival rate r_{max}. The delay optimal retransmission probability lies in the feasible region in the interior of the parabola.

4 Analytical Insights

In this section, we consider various capture scenarios and compare the delay performance of ALOHA.

4.1 Maximum stable throughput and Critical rate

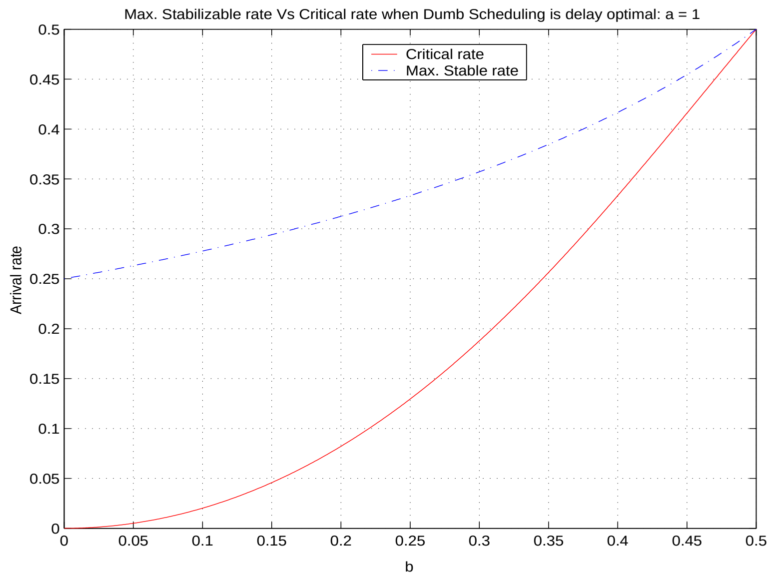

Figure 3: Maximum stable throughput vs critical rate $a = 1$

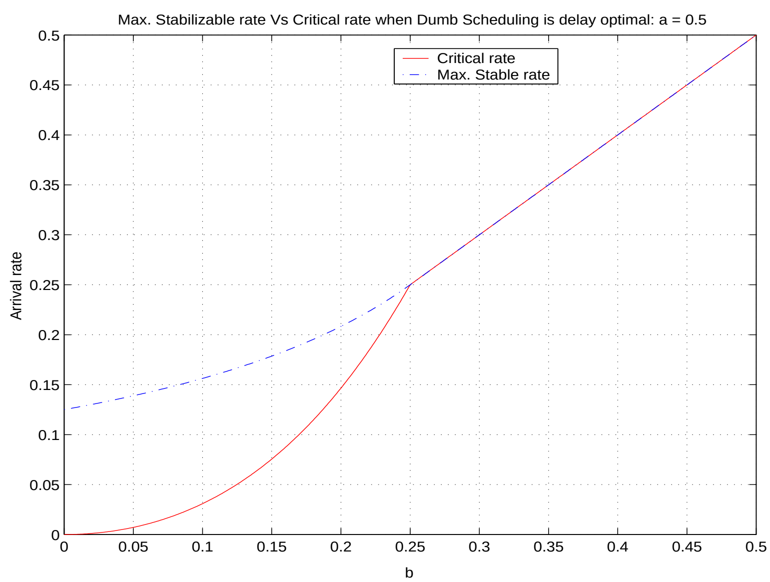

Figure 4: Maximum stable throughput vs critical rate $a = 0.5$

Figures 3 and 4 compare the critical rate with maximum stable arrival rate for all possible capture scenarios. We see a phase transition here that occurs at the point $b = a/2$. As long as $b < a/2$, dumb scheduling is only optimal for a subset of the stabilizable rates. On the other hand as soon as $b \geq a/2$, dumb scheduling is optimal for all stabilizable rates. Note that all rates below the solid curve are delay optimized by dumb scheduling.

4.2 Delay comparison of different capture channels

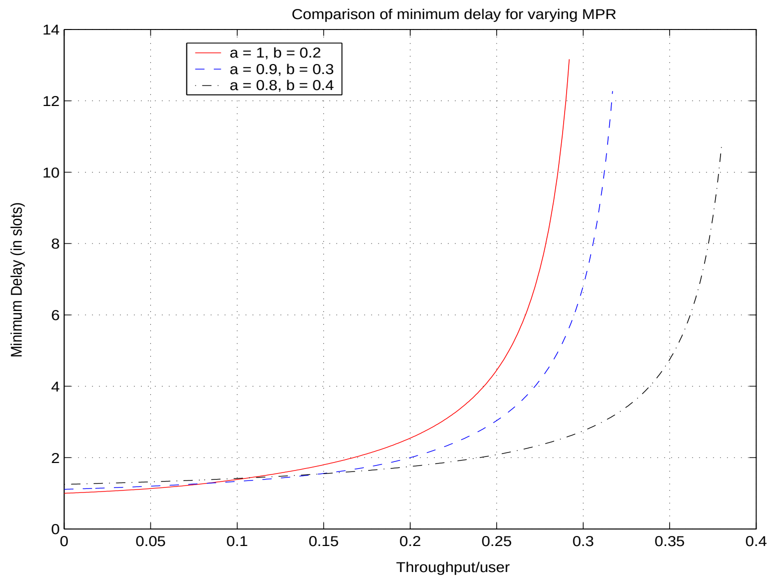

Figure 5: Comparison of delay for various capture scenarios, $a = 1, 0.9, 0.8$, $b = 0.2, 0.3, 0.4$

Figure 5 compares the minimal delay for three capture scenarios. In this case, we increase a and decrease b progressively. It can be seen that at low arrival rates the capture model with $a = 1$, $b = 0.2$ is marginally better that the other capture models. At higher arrival rates, the capture model with $a = 0.8$, $b = 0.4$ is significantly better than the others. Thus, it seems that for minimizing delay, "multiuser" receiver design is much better than the omnipresent "single user" designs.

Figure 6 compares the minimal delay in collision channel $a = 1$, $b = 0$ with the delay in strong capture scenarios. It illustrates the significant average delay reduction that can be achieved with the strongest capture model $a = 1$, $b = 0.5$. We also note that the minimal delay in this strong capture model ($a = 1$, $b = 0.5$) is quite close to one for arrival rates until ~ 0.25. Since the average delay is lower bounded by one, this suggests that ALOHA is quite close to optimal for a large class of arrival rates for strong capture models.

4.3 Delay comparison with fixed "a"(or "b")

Figures 7 and 8 show minimal delay as a function of the arrival rate for fixed a and fixed b respectively. In Figure 7 the curves are far apart as compared to those in Figure 8. The figures show that the delay is much more sensitive to changes in b than a.

At this point it is not clear what would happen if we had a stronger reception model than the capture model we have considered in this work. First of all, the technique used to find the average queue length fails as terms corresponding to probability of success of both

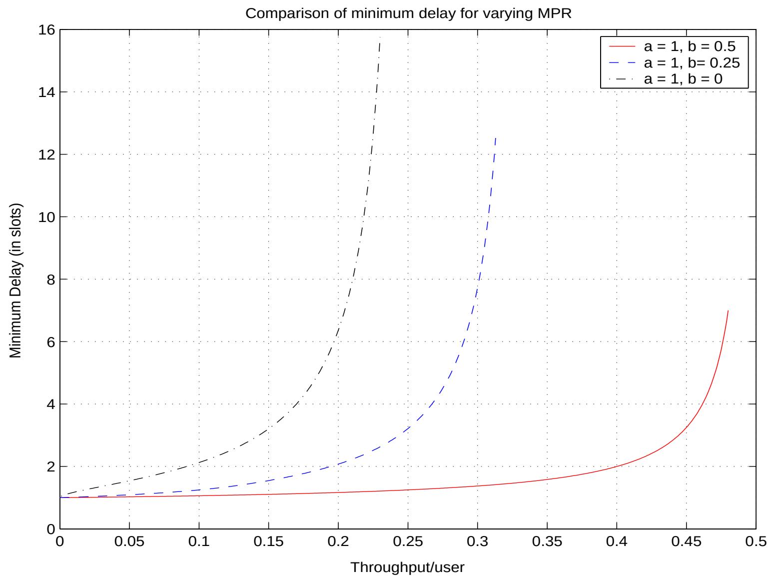

Figure 6: Comparison of delay for capture scenarios with the collision channel, $a = 1, 1, 1$, $b = 0.5, 0.25, 0$

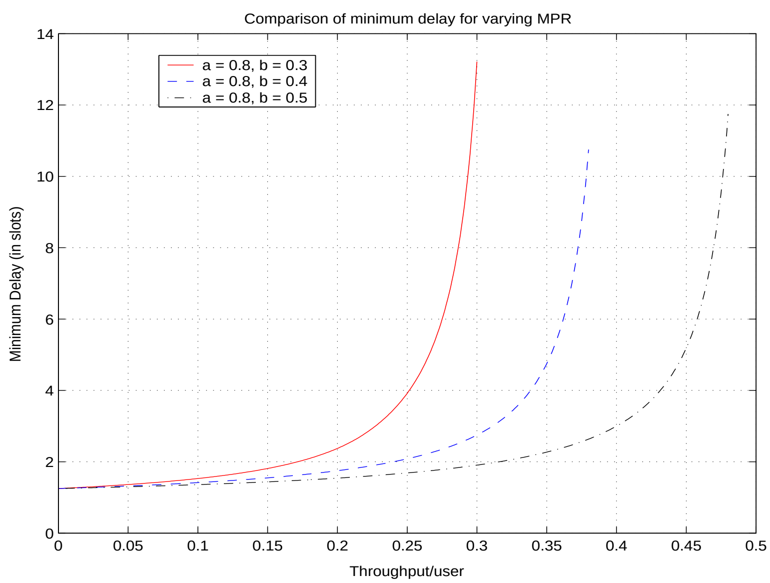

Figure 7: Comparison of delay for various capture scenarios, $a = 0.8, 0.8, 0.8$, $b = 0.3, 0.4, 0.5$

users *simultaneously* lead to some complications. However, we conjecture that even with a stronger (symmetric) reception model that allows simultaneous packet receptions from both users with marginal success probabilities greater than 0.5, $p = 1$ would minimize the delay. The intuition behind this belief is that there is no reason for the users to hold back transmissions in the presence of a stronger reception model.

We also suspect that the delay results that we have will be upper bounds on delay of more general Multipacket Reception models which have the same marginal reception probabilities as the capture model we consider.

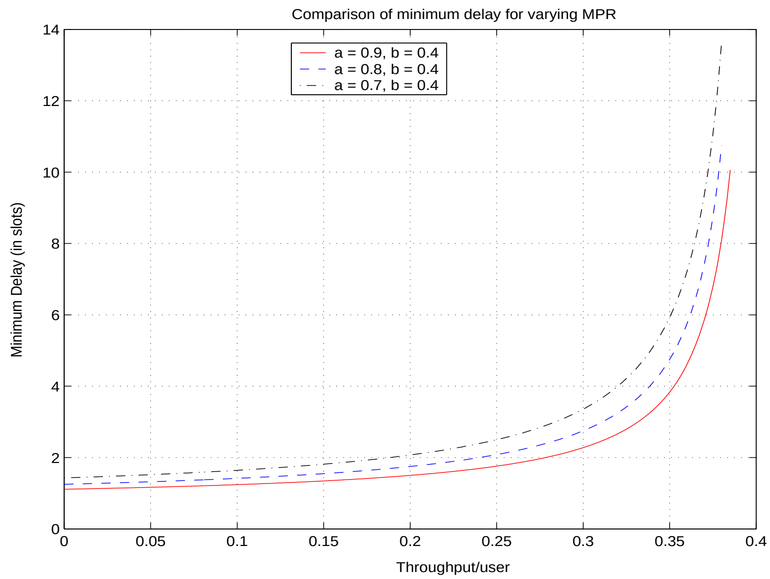

Figure 8: Comparison of delay for various capture scenarios, $a = 0.9, 0.8, 0.7$, $b = 0.4, 0.4, 0.4$

5 Conclusions

In this work, we analyzed the delay performance of slotted ALOHA in a capture environment albeit for the two user case. We provided expressions for the delay optimizing retransmission probability and exact average delay of slotted ALOHA in capture channels. We showed that dumb scheduling is delay optimal for all stable arrival rates in a certain capture regime and that dumb scheduling is always optimal for a subset of stable arrival rates once capture sets in. It is interesting to note that when dumb scheduling is delay optimal for all stable arrival rates, it also stabilizes all stable arrival rates.

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