Joint Frequency and Phasor Estimation Under the KCL Constraint

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***Abstract*—In this letter, we consider the problem of joint off-nominal frequency and phasor estimation that incorporates Kirchhoff's Current Law (KCL) as a constraint. We develop the constrained maximum likelihood (CML) and constrained weighted least-squares (CWLS) estimators for this problem and derive the corresponding constrained Cramér–Rao bound. The KCL constraint is shown to behave as a noise cancellation factor for the phasors estimation. We show that the KCL CML is based on the classical periodogram subtracting the average current periodogram. The results indicate significant performance improvement compared to the unconstrained maximum likelihood (ML) and unconstrained weighted least-squares (WLS).**

***Index Terms*—Constrained Cramér–Rao bound, constrained maximum likelihood estimation, frequency estimation, phasor measurement unit (PMU), power system state estimation.**

I. Introduction

In this letter we consider a frequency estimation problem in the context of power system state estimation using phasor measurement units (PMUs). A key feature of PMU is its ability to capture frequency and phasor changes at a much higher resolution than traditional meters period. PMU provides synchronized direct measurements of bus voltages and currents. Under normal circumstances a power system operates at a nominal frequency with small, but time varying, frequency-deviation [1], [2]. In events when system contingencies arise, however, substantial frequency-deviation may occur, and the state of power system can change rapidly. The ability to estimate and track varying frequency-deviation is highly desirable, and simple techniques based on discrete Fourier transform may be insufficient. In the literature, various frequency estimation methods for power system were proposed under both balanced and unbalanced power systems (e.g., [3], [4]).

Frequency and phasor estimation in power systems, from a signal processing viewpoint, fall in the category of classical frequency estimation [5]. What differentiates the problem from the standard formulation is that the current measurements satisfy Kirchhoff's Current Law (KCL) at each bus. The KCL constraint is used in [6] for correcting current measurements by employing ℓ_1-norm minimization for nominal-frequency power systems. However, frequency estimation under such a constraint has not been previously studied and has the potential of improving estimation accuracy.

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We propose two methods for joint system frequency and phasor estimation by incorporating KCL in the estimator. The problem is formulated as one of constrained maximum likelihood (CML) estimation [7], [8]. The resulting frequency estimator is similar to the classical periodogram estimator but with a modification involving periodograms of current and average current signals. This modification plays the role of noise cancellation and side lobe attenuation thus, providing improved estimation performance. In addition, we describe how any suboptimal method can be improved by enforcing KCL and we also describe a particular example of constrained weighted least-squares (CWLS) estimation. We develop the appropriate constrained Cramér–Rao bound (CCRB) [8] for this case, and we show that the CCRB of the phasors estimation is lower than the unconstrained Cramér–Rao bound (CRB) while the CCRB of the frequency estimation remains the same as the CRB [5]. The simulation results demonstrate that KCL information significantly improves the phasors estimation in terms of mean-square-error (MSE) reduction.

II. Phasor and Frequency Estimation With KCL

A. Problem Formulation

We consider the voltage and current measurements by a PMU at a specific bus. In particular, the PMU measures synchronously the voltage at the bus and current flows on lines incident to the bus, as presented schematically in Fig. 1. The complex representation of these signals at off-nominal frequency $\omega_0 + \Delta$ is given by [1]:

$$\begin{cases} i_m(t) = \frac{X_m}{\sqrt{2}} e^{j(\omega_0+\Delta)t}, & \forall m = 1,\dots,M \\ v(t) = \frac{X_{M+1}}{\sqrt{2}} e^{j(\omega_0+\Delta)t} \end{cases}, \qquad (1)$$

where ω_0 is the (known) nominal frequency (100π or 120π), Δ is the frequency-deviation from this nominal value, and $X_m, m = 1,\dots,M$ and X_{M+1} denote the currents and voltage phasors, respectively, where the phasors belong to the off-nominal frequency, $\omega_0+\Delta$. Using KCL, we have $\sum_{m=1}^{M} i_m(t) = 0$.

Assuming that $\{i_m(t)\}_{m=1}^{M}$ and $v(t)$ are sampled N times per cycle of the nominal frequency to produce the following noisy discrete time measurements model:

$$x_m[n] = \frac{X_m}{\sqrt{2}} e^{j\gamma \frac{\omega_0+\Delta}{\omega_0} n} + w_m[n], m = 1,\dots,M+1, \quad (2)$$

where $x_m[n], n = 0,\dots,N-1$ are samples corresponding to the mth current signal, $i_m[t]$, for $1 \leq m \leq M$, $x_{M+1}[n], n = 0,\dots,N-1$ are the voltage samples, $\gamma \triangleq (2\pi)/(N)$ is the sampling angle, and $\{w_m[n]\}_{n=0}^{N-1}$, $m = 1,\dots,M+1$ are independent complex circularly symmetric white Gaussian noise sequences with known variances σ_m^2. The unknown real parameter vector is $\boldsymbol{\theta} = [|X_1|,\dots,|X_{M+1}|, \sphericalangle(X_1),\dots,\sphericalangle(X_{M+1}),\Delta]^{\mathrm{T}}$, where $\sphericalangle(\cdot)$ denotes the angle of its argument and $X_m = |X_m| e^{j\sphericalangle(X_1)}$,

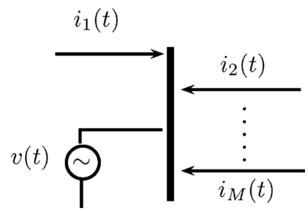

Fig. 1. Single PMU model at one node with M currents and voltage measurements.

for any $m = 1, \ldots, M+1$. In order to avoid ambiguities, we restrict the estimates of Δ to $[-(N\omega_0)/(2), (N\omega_0)/(2))$.

In this work, we are interested in the joint estimation of the phasors and frequency-deviation where KCL enforces the constraint

$$\sum_{m=1}^{M} X_m = 0. \tag{3}$$

B. The CML Estimation

The following shows how to include KCL *a-priori* information in the maximum likelihood (ML) estimate in order to improve the estimation and achieve a lower MSE. For the specified PMU measurement model in (2), the log-likelihood function is given by

$$L(\boldsymbol{\theta}) = C - \sum_{m=1}^{M+1} \frac{1}{\sigma_m^2} \sum_{n=0}^{N-1} \left| x_m[n] - \frac{1}{\sqrt{2}} X_m e^{j\gamma \frac{\omega_0+\Delta}{\omega_0} n} \right|^2, \tag{4}$$

where C is a constant independent of $\boldsymbol{\theta}$. Incorporating the KCL constraint from (3), the CML estimate of $\boldsymbol{\theta}$ is given by

$$\min_{\boldsymbol{\theta}} L(\boldsymbol{\theta}) \text{ subject to } \sum_{m=1}^{M} X_m = 0. \tag{5}$$

In the Appendix, it is proved that minimization in (5) results in the CML estimators described in the following theorem.

Theorem 1: Let the summed current signal be

$$\bar{i}[n] = \sum_{m=1}^{M} x_m[n], \quad \forall n = 0, 1, \ldots, \tag{6}$$

the current/voltage and summed current periodograms are

$$\begin{cases} P_m(\alpha) = \frac{1}{N} \sum_{n=0}^{N-1} x_m[n] e^{-j\gamma n \frac{\omega_0+\alpha}{\omega_0}}, m = 1, \ldots, M+1 \\ \bar{P}(\alpha) = \frac{1}{N} \sum_{n=0}^{N-1} \bar{i}[n] e^{-j\gamma n \frac{\omega_0+\alpha}{\omega_0}} \end{cases}$$

and defined the following current transformation

$$\tilde{x}_m[n] = \begin{cases} x_m[n] - \frac{\sigma_m^2}{\sum_{l=1}^{M} \sigma_l^2} \bar{i}[n] & m = 1, \ldots, M \\ x_m[n] & m = M+1 \end{cases}, \tag{7}$$

$\forall n = 0, \ldots, N-1$. Then, the CML estimators of the frequency-deviation and phasors of the different currents and voltage are given by:

$$\hat{\Delta} = \arg \max_{\alpha \in [-\frac{N\omega_0}{2}, \frac{N\omega_0}{2})} \sum_{m=1}^{M+1} \frac{|P_m(\alpha)|^2}{\sigma_m^2} - \frac{|\bar{P}(\alpha)|^2}{\sum_{l=1}^{M} \sigma_l^2}, \tag{8}$$

$$\hat{X}_m = \frac{\sqrt{2}}{N} \sum_{n=0}^{N-1} \tilde{x}_m[n] e^{-j\gamma n \frac{\omega_0+\hat{\Delta}}{\omega_0}}, \quad \forall m = 1, \ldots, M+1. \tag{9}$$

For the sake of simplicity, we derive the CML estimators of the complex phasors instead of the real amplitudes and phases in $\boldsymbol{\theta}$. However, by using the invariance property of the ML estimator, the CML estimators of these parameter can be obtain directly from (9) by taking the magnitudes and the phases of the estimated phasors.

In practice, the maximization in (8) may be approximated by finding the optimal discrete frequency [5]. This method involves a quantization error because it only evaluates the objective function at discrete frequencies.

C. Relation to the Unconstrained ML Estimation

Similar to the derivation in the Appendix, the unconstrained ML estimators are [5]

$$\hat{\Delta}^{(\mathrm{uc})} = \arg \max_{\alpha \in [-\frac{N\omega_0}{2}, \frac{N\omega_0}{2})} \sum_{m=1}^{M+1} \frac{1}{\sigma_m^2} |P_m(\alpha)|^2, \tag{10}$$

$$\hat{X}_m^{(\mathrm{uc})} = \frac{\sqrt{2}}{N} \sum_{n=0}^{N-1} x_m[n] e^{-j\gamma n \frac{\omega_0+\hat{\Delta}^{(\mathrm{uc})}}{\omega_0}}, \tag{11}$$

$\forall m = 1, \ldots, M+1$. Thus, the KCL information adds correction terms to the frequency-deviation and current phasors estimators and the phasors estimators are affected by the KCL throughout $\hat{\Delta}$. It can be verified that the CML estimators in (8)–(9) can be obtained directly by using the unconstrained ML estimation method in (10)–(11) with the transformed measurements in (7).

D. Noise Cancellation

By using the model in (2) and (6), it can be observed that

$$\bar{i}[n] = \sum_{m=1}^{M} X_m e^{j\gamma \frac{\omega_0+\Delta}{\omega_0} n} + \sum_{m=1}^{M} w_m[n]. \tag{12}$$

Substitution of the KCL constraint from (3) in (12), results in

$$\bar{i}[n] = \sum_{m=1}^{M} w_m[n]. \tag{13}$$

Therefore, the summed current signal, $\bar{i}[n]$, is in fact a noise signal and is used in (7) as a noise cancellation term. The term $x_m[n] - (\sigma_m^2)/(\sum_{l=1}^{M} \sigma_l^2)\bar{i}[n]$ in (7) is the branch phasor minus the weighted sum of all the branch noises. The weighting is done according to (known) noise variances.

The frequency-deviation estimator in (8) can be interpreted as the frequency that maximizes the weighted average of the periodograms of the currents and the voltages, $P_1(\alpha), \ldots, P_{M+1}(\alpha)$, minus the periodogram of the summed current signal, $\bar{P}(\alpha)$. Observing (13), we note that $\bar{P}(\alpha)$ is a periodogram of the average noise. For low signal-to-noise ratios (SNR), noise artifacts appear in the ML spectral estimates based on the periodogram. The CML reduces these artifacts and improves the estimate. However, the white noise periodogram is a constant for high SNRs, thus, asymptotically it does not have an effect on the maximum value.

III. Low-Complexity Estimation Methods

In practice, since the ML estimator of the frequency-deviation in (8) requires search procedure and suffers from high complexity, many other low-complexity frequency estimation methods are used in power systems [1]. In order to enforce the KCL in these suboptimal methods, one can change the original measurements in (2) to the corrected measurements in (7). Since the corrected measurements have a higher SNR compared to the original measurements, the suboptimal methods with the corrected measurements are expected to result in estimators of greater accuracy with a lower MSE. This procedure is exemplified here on the weighted least-squares (WLS) method.

According to (2), the angle measurements are given by (e.g., Chapter 4 in [1]) $\boldsymbol{\phi} = \mathbf{B}\boldsymbol{\theta}_{M+2:2M+3}$, where

$$\boldsymbol{\phi} \triangleq \Big[\sphericalangle(x_1[0]), \ldots, \sphericalangle(x_{M+1}[0]), \sphericalangle(x_1[1]^{-j\gamma}), \ldots, \sphericalangle(x_{M+1}[1]^{-j\gamma}), \ldots, \sphericalangle\left(x_{M+1}[N-1]^{-j\gamma(N-1)}\right)\Big]^T,$$

$\boldsymbol{\theta}_{M+2:2M+3}$ includes the last $M+2$ elements of $\boldsymbol{\theta}$, and

$$\mathbf{B} = \begin{bmatrix} \mathbf{I}_{M+1} & \mathbf{0}_{M+1,1} \\ \mathbf{I}_{M+1} & \gamma/\omega_0 \mathbf{1}_{M+1,1} \\ \vdots & \vdots \\ \mathbf{I}_{M+1} & \gamma(N-1)/\omega_0 \mathbf{1}_{M+1,1} \end{bmatrix}.$$

Therefore, the WLS estimator that minimizes $\|\boldsymbol{\phi} - \mathbf{B}\mathbf{a}\|^2_{\mathbf{W}}$, is

$$\hat{\boldsymbol{\theta}}^{(\mathrm{WLS})}_{M+2:2M+3} = (\mathbf{B}^T\mathbf{W}^{-1}\mathbf{B})^{-1}\mathbf{B}^T\mathbf{W}^{-1}\boldsymbol{\phi}, \quad (14)$$

where the wighted matrix is $\mathbf{W} = \mathrm{Diag}(\sigma_1^2, \ldots, \sigma_{M+1}^2, \sigma_1^2, \ldots, \sigma_{M+1}^2, \ldots, \sigma_{M+1}^2)$.
For example, if $\sigma_1^2 = \sigma_2^2 = \cdots = \sigma_{M+1}^2$ the estimators in (14) are reduced to

$$\sphericalangle(\hat{X}_m^{\mathrm{WLS}}) = \frac{1}{N}\sum_{n=0}^{N-1}\left(\sphericalangle(x_m[n]) - \frac{\gamma}{\omega_0}n\hat{\Delta}^{\mathrm{WLS}}\right)$$

$$\hat{\Delta}^{\mathrm{WLS}} = c\sum_{m=1}^{M+1}\sum_{n=0}^{N-1}\left(n - \frac{N-1}{2}\right)\sphericalangle(x_m[n]),$$

where $c \triangleq (12\omega_0)/(\gamma N(M+1)(N^3 - N))$. Finally, the magnitudes WLS estimators are calculated by

$$\left|\hat{X}_m^{(\mathrm{WLS})}\right| = \frac{\sqrt{2}}{N}\left|\sum_{n=0}^{N-1} x_m[n]e^{-j\gamma n\frac{\omega_0 + \hat{\Delta}^{(WLS)}}{\omega_0}}\right|, \quad (15)$$

$\forall m = 1, \ldots, M+1$.

The CWLS estimation under the KCL constraint is obtained by substituting the corrected measurements $\{\tilde{x}_m[n]\}$ from (7) in (14) and (15) instead of the original measurements $\{x_m[n]\}$.

IV. Constrained Cramér–Rao Bound

In this section we derive the CCRB for jointly frequency and phasor estimation under the KCL constraint. The elements of the Fisher information matrix (FIM) for the measurements model in (2) are [5]

$$\mathbf{J}_{k,m} = 2\sum_{n=0}^{N-1}\mathrm{Re}\left\{\frac{\partial\boldsymbol{\mu}^H[n]}{\partial\theta_k}\boldsymbol{\Sigma}^{-1}\frac{\partial\boldsymbol{\mu}[n]}{\partial\theta_m}\right\}, \quad (16)$$

$\forall k, m = 1, \ldots, 2M+3$, where

$$\boldsymbol{\mu}[n] = \frac{1}{\sqrt{2}}\left[X_1 e^{j\gamma\frac{\omega_0+\Delta}{\omega_0}n}, \ldots, X_{M+1}e^{j\gamma\frac{\omega_0+\Delta}{\omega_0}n}\right]^T$$

and $\boldsymbol{\Sigma} = \mathrm{Diag}(\sigma_1^2, \ldots, \sigma_M^2, \sigma_{M+1}^2)$. Using (16), and the block-inverse of matrices, the CRB for this case is

$$\mathrm{CRB} = \begin{bmatrix} \boldsymbol{\Sigma} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_2^{-1} - \frac{\gamma^2(N-1)^2}{4\omega_0^2 a}\mathbf{1}\mathbf{1}^T & -\frac{\gamma(N-1)}{2\omega_0 a}\mathbf{1} \\ \mathbf{0} & -\frac{\gamma(N-1)}{2\omega_0 a}\mathbf{1}^T & a^{-1} \end{bmatrix}, \quad (17)$$

where $\mathbf{1}$ is $M+1$ vector of ones,

$$a = J_3 - \mathbf{J}_{2,3}^T\mathbf{J}_2^{-1}\mathbf{J}_{2,3} = \frac{\gamma^2 N(N^2-1)}{12\omega_0^2}\sum_{m=1}^{M+1}\frac{|X_m|^2}{\sigma_m^2},$$

in which

$$\mathbf{J}_2 = N\mathrm{Diag}\left(|X_1|^2/\sigma_1^2, \ldots, |X_{M+1}|^2/\sigma_{M+1}^2\right),$$

$$\mathbf{J}_{2,3} = \frac{\gamma N(N-1)}{2\omega_0}\left[|X_1|^2/\sigma_1^2, \ldots, |X_{M+1}|^2/\sigma_{M+1}^2\right]^T,$$

$$J_3 = \frac{\gamma^2 N(N-1)(2N-1)}{6\omega_0^2}\sum_{m=1}^{M+1}\frac{|X_m|^2}{\sigma_m^2}.$$

The KCL constraint in (3) can be rewritten as

$$\mathbf{f}(\boldsymbol{\theta}) = \begin{bmatrix} \sum_{m=1}^M |X_m|\cos(\sphericalangle(X_m)) \\ \sum_{m=1}^M |X_m|\sin(\sphericalangle(X_m)) \end{bmatrix} = \mathbf{0}. \quad (18)$$

Thus, the gradient matrix, $\mathbf{F} = (\partial f(\boldsymbol{\theta}))/(\partial\boldsymbol{\theta})$, is given by

$$\mathbf{F} = \begin{bmatrix} \cos(\sphericalangle(X_1)) & \ldots & \cos(\sphericalangle(X_M)) & 0 & -|X_1|\sin(\sphericalangle(X_1)) & \ldots & -|X_M|\sin(\sphericalangle(X_M)) & 0 & 0 \\ \sin(\sphericalangle(X_1)) & \ldots & \sin(\sphericalangle(X_M)) & 0 & |X_1|\cos(\sphericalangle(X_1)) & \ldots & |X_M|\cos(\sphericalangle(X_M)) & 0 & 0 \end{bmatrix}. \quad (19)$$

The CCRB for nonsingular FIM is given by [8]

$$\mathrm{CCRB} = \mathbf{J}^{-1} - \mathbf{J}^{-1}\mathbf{F}^T(\mathbf{F}\mathbf{J}^{-1}\mathbf{F}^T)^{-1}\mathbf{F}\mathbf{J}^{-1}. \quad (20)$$

By substituting (17) and (19) in (20), we obtain the CCRB for this case.

The last row of the matrix $\mathbf{J}^{-1}\mathbf{F}^{\mathrm{T}}$ is

$$\left[a^{-1}\sum_{m=1}^M |X_m|\sin(\sphericalangle(X_m)), \; -a^{-1}\sum_{m=1}^M |X_m|\cos(\sphericalangle(X_m))\right],$$

which is equal to $[0, 0]$ according to the KCL constraint in (18). The CCRB in (20) implies that the scalar constrained bound on the frequency-deviation MSE is

$$[\mathrm{CCRB}]_{2M+3,2M+3} = [\mathrm{CRB}]_{2M+3,2M+3} = a^{-1}. \quad (21)$$

Thus, the KCL constraint has no influence on the CRB of the frequency-deviation, which indicates that the *asymptotic* performance of the ML and CML frequency estimators is identical. However, in the non-asymptotic region the ML and CML frequency estimators are different. This phenomenon is also demonstrated in Subsection II.D and in the simulations.

The block-diagonal structure of the CRB and CCRB in (17) and (20), respectively, implies that the unconstrained phasors estimators are decoupled from each other while the constrained estimators are conjugated. The magnitudes are decoupled from

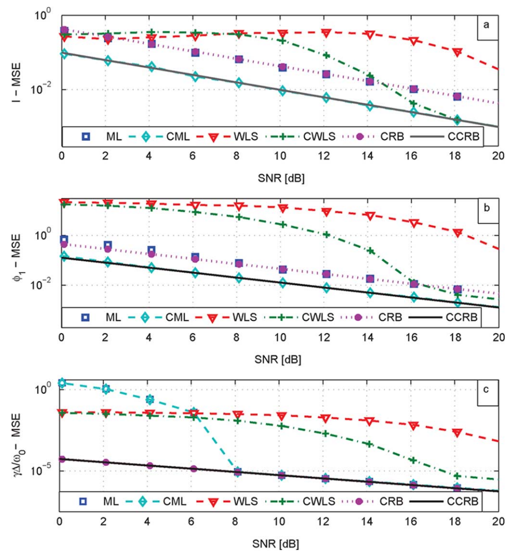

Fig. 2. The CRB, CCRB, and the MSE of the ML, CML, WLS, and CWLS estimators of $|X_1|$ (a), $\sphericalangle(X_1)$ (b), and $(\gamma\Delta)/(\omega_0)$ (c).

the phases and the frequency-deviation for both the unconstrained and unconstrained setting.

V. Simulations

In this section, we compare the performance of the ML, CML, WLS, and CWLS methods. We consider single PMU with $M = 4$ branches of equal magnitudes. The frequency-deviation is set to be $\Delta = 2\pi \cdot 5$ where the nominal-frequency is $\omega_0 = 2\pi \cdot 60$. The sampling rate is 48 samples per cycle of the nominal power frequency. The SNRs satisfy $(|X_1|^2)/(10\sigma_1^2) = (|X_2|^2)/(\sigma_2^2) = (|X_3|^2)/(\sigma_3^2) = (|X_4|^2)/(\sigma_4^2) = (|X_5|^2)/(\sigma_5^2)$. The performance were evaluated using 5000 Monte-Carlo simulations.

The MSE of the ML, CML, WLS, and CWLS estimators of the current magnitudes and phases at branch 1, $\mathrm{E}[(|\hat{X}_1| - |X_1|)^2]$ and $\mathrm{E}[(\sphericalangle(\hat{X}_1) - \sphericalangle(X_1))^2]$, respectively, and the frequency-deviation $(\gamma\Delta)/(\omega_0)$ are presented in Fig. 2 and compared to the CRB and CCRB.

Figs. 2(a) and (b) show that the MSE of the ML (or WLS) estimator is significantly higher than the MSE of the CML (or CWLS) estimator for both the magnitude and phase. In addition, the suboptimal CWLS estimator has a lower MSE than the ML estimator without constraint. Fig. 2(c) demonstrates that although the KCL constraint has a small effect on the frequency-deviation in this case, as expected by the CRB and CCRB, it significantly improves the performance of the suboptimal WLS estimation.

VI. Conclusion

In this letter we have demonstrated the possibility of achieving parameter estimation of greater accuracy by using KCL. The CML and suboptimal CWLS are derived for frequency-deviation and phasors estimation based on PMU measurements and KCL. Topics for future research include the derivation of estimation methods for multiple frequencies, estimation with several PMUs in a network, and incorporation of the positive and negative sequences of three-phase voltage and currents samples similar to [9].

Appendix
Proof of Theorem 1

By using Lagrange multipliers, the CML estimation in (5) is the minimum of the following Lagrangian

$$Q = L(\boldsymbol{\theta}) + \lambda \sum_{m=1}^{M} X_m \tag{22}$$

where λ is the Lagrange multiplier.

Phasors estimation: For a fixed Δ, by equating the complex derivatives of (22) with respect to the complex function X_l, $(\partial Q)/(\partial X_l)$, to zero, one obtains

$$0 = \frac{1}{\sqrt{2}\sigma_l^2} \sum_{n=0}^{N-1} x_l^*[n] e^{j\gamma \frac{\omega_0+\Delta}{\omega_0} n} - \frac{N}{2\sigma_l^2} \hat{X}_l^* + \lambda, \tag{23}$$

$\forall l = 1, \ldots, M$. By summing (23) over $l = 1, \ldots, M$, one obtains

$$\frac{1}{\sqrt{2}} \sum_{l=1}^{M} \sum_{n=0}^{N-1} x_l^*[n] e^{j\gamma \frac{\omega_0+\Delta}{\omega_0} n} - \frac{N}{2} \sum_{l=1}^{M} \hat{X}_l^* = -\lambda \sum_{l=1}^{M} \sigma_l^2. \tag{24}$$

Substitution of the constraint from (3) in (24) yields

$$\lambda = -\frac{1}{\sqrt{2}\sum_{l=1}^{M}\sigma_l^2} \sum_{l=1}^{M} \sum_{n=0}^{N-1} x_l^*[n] e^{j\gamma \frac{\omega_0+\Delta}{\omega_0} n}. \tag{25}$$

Substitution of (25) in (23) and using (7), results in (9) $\forall m = 1, \ldots, M$. Similarly, we obtain (9) for $m = M + 1$.

Frequency estimation: By substituting $X_m = \hat{X}_m$ and $\sum_{m=1}^{M} \hat{X}_m = 0$ in (22), one obtains the following cost

$$Q = -\sum_{m=1}^{M+1} \frac{1}{\sigma_m^2} \sum_{n=0}^{N-1} |x_m[n]|^2 + \sum_{m=1}^{M+1} \frac{N}{2\sigma_m^2} |\hat{X}_m|^2. \tag{26}$$

The left term in (26) is independent of Δ, thus, by using (9) it can be seen that the frequency-deviation estimator can be rewritten in terms of periodograms as in (8).

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