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Stability and Capacity of Wireless Networks with Probabilistic Receptions

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Abstract

We study the stability and the capacity problems in packetized wireless networks. Communication medium is modelled using probability density functions that determine the packet reception probabilities. The model subsumes several previous models as special cases, and it is suitable for networks with time-varying topology and channels. Our main result is a characterization of the stability and the capacity regions using network flows. We also introduce a class of control policies sufficient to achieve every rate inside these regions. In the second part of the paper, we apply the proposed policies and the flow analysis to regular networks. We obtain closed-form expressions for the capacity of Manhattan networks (two-dimensional grid) and ring networks (circular array of nodes). We analyze the performance loss due to suboptimal medium access and routing. We also investigate the impact of link fading, link state information, and variable connectivity on achievable rates in Manhattan networks.

1 Introduction

The objective of this work is twofold. First, we provide a general approach to characterizing the capacity and stability regions of networks with a probabilistic reception model. This model, defined by the conditional probability of successful receptions given the subset of transmitting users, is sufficiently general to include multipacket receptions and links with ergodic fading. Second, we aim to provide insights and design guidelines by examining the class of one-dimensional (ring) and two-dimensional (Manhattan) regular networks. Having obtained closed-form expressions for the capacity, we are able to quantify the loss incurred by suboptimal protocols, the gain obtained by using link state information, and the effects of increasing connectivity.

1.1 General Results on Stability and Capacity

The network capacity problem deals with finding the fundamental limits on achievable communication rates in wireless networks. A set of rates between source-destination pairs is called achievable if there exists a network control policy that guarantee those rates. The closure of the set of achievable rates is the *capacity region* of the network. Our main result is a characterization of the capacity region using network flows. In the flow characterization one needs to assign a probability density over the set of transmission schedules for medium access (MAC). Similarly, routing amounts to assigning a probability density over the set of routes. We show that all rates inside the capacity region can be represented as a flow feasible with certain probability densities for MAC and routing. To establish this result we introduce a class of control policies that do randomized routing and medium access. These will be called *randomized time-division* (RTD) policies since their MAC can be viewed as a randomized version of time-division multiple access (TDMA).

In the capacity analysis the notion of *transport capacity* plays an important role. The transport capacity introduced by Gupta and Kumar [1] measures the delivery rate times the

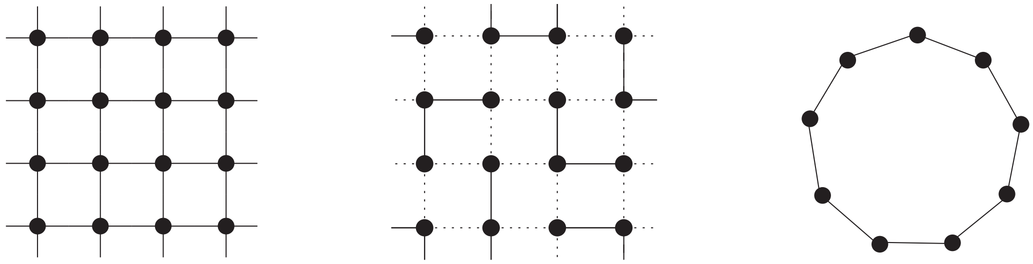

Figure 1: Examples of regular networks

distance packets travel. In this paper, we extend the definition of transport capacity to networks with probabilistic receptions. This extension allows us to handle the transport capacity in a more general setting where the network possibly has time variation and the distance metric is not Euclidean. The generalized transport capacity is used extensively in proving upper bounds on the capacities of regular networks.

In the capacity analysis, it is assumed that there always are packets to be delivered at the source nodes. However, in reality the data packets arrive randomly in time, and for proper network operation the node buffers should be kept stable. Intuitively, the network is called stable if the node buffers do not overflow during operation. The *stability region* is the closure of the set of arrival rates at which the network can be stabilized using a control policy. Stability depends both on the rate of packet arrivals and the rate of packet departure from the network. The latter is of course closely related to the notion of achievability. One would expect that the stability region must be inside the capacity region; this is true, since the delivery rate is equal to the arrival rate in stable networks. The converse, however, is false in general; some part of the capacity region may lie outside the stability region in networks with probabilistic receptions. In Section 2 we provide some mild conditions on the network under which the stability and the capacity regions are the same. In proving this result, we again exploit the flow characterization and the RTD policies. In particular, we show that for every arrival rate inside the stability region there exists an RTD policy stabilizing the network.

1.2 Capacity of Regular Networks

In the second part of the paper, we apply the developed theory for obtaining the network capacity and optimal RTD policies for regular networks (*e.g.,* see Figure 1). Although it is hardly the case that networks in practice are regular, the regular networks provide valuable insights by being amenable to analysis. Our main focus is the Manhattan network which is a two-dimensional grid with size $\sqrt{N} \times \sqrt{N}$. In the Manhattan network every node has four neighbors, and the nodes on the edge are connected to the nodes on the opposite edge, forming a torus. In Manhattan networks, we use the multipacket reception (MPR) channel model [2], [3] which allows simultaneous multiple receptions and random channel errors.

We first show that the capacity of a Manhattan network is

$$\eta = \frac{K_1}{\sqrt{N}} + O(\frac{1}{N}),$$

where the coefficient K_1 (given in subsection 6.2) depends only on the channel reception capability.[1] In case nodes can simultaneously receive multiple packets, K_1 increases but the form of η does not change.

Even though the capacity η can be achieved by using optimal medium access and routing, it is important to quantify the loss because of using suboptimal, yet more practical, control policies. We will look at two extremes: a simple medium access method (slotted ALOHA), and a simple routing (random walking packets). We show that the maximum achievable rate with slotted ALOHA medium access and optimal routing is

$$\eta_{ALOHA} = \frac{K_2}{\sqrt{N}} + O(\frac{1}{N}),$$

where the coefficient K_2 (also, given in subsection 6.2) is smaller than K_1. On the other hand, the achievable rate with optimal medium access and random walking packets is $O(\frac{1}{N \log N})$. These results suggest that the medium access method in general does not change the order of the capacity, but the routing does change the order, and a poor routing protocol can significantly degrade the performance of large networks.

We next consider the case where the links in Manhattan network are subject to time-varying fading. We use the collision channel model with a simple model for fading; links become ON/OFF randomly in each slot (ON with probability p, OFF with probability $1-p$). A realization of this network is depicted in Figure 1.b, where the OFF links are shown with dashed lines. In case the control policy doesn't know the states of the links before making transmission decisions, we say that the policy is without link state information (LSI). The capacity *without* LSI is shown to be

$$\eta = \frac{K_3}{\sqrt{N}} + O(\frac{1}{N}),$$

where the coefficient K_3 is given in subsection 6.4. We will develop upper bounds on the capacity *with* LSI. Let $\eta^{\#}$ be the capacity with LSI. We show that the ratio $\eta^{\#}/\eta$ satisfies

$$1 \leq \frac{\eta^{\#}}{\eta} \leq 2.86 + O(1/\sqrt{N}).$$

Furthermore, the bounds for $\eta^{\#}/\eta$ are not very loose, that is, $\eta^{\#}/\eta$ is equal to 2.5 in the limit as $p \rightarrow 0,\ N \rightarrow \infty$. Similarly, $\eta^{\#}/\eta$ is equal to 1 in the limit as $p \rightarrow 1,\ N \rightarrow \infty$.

Finally, we look at the optimal network-connectivity problem. Gupta and Kumar [1], and Gallager and Bertsekas [4, p. 350] discussed the trade-off between throughput vs. connectivity and argued that minimizing transmission radius while keeping the network connected leads to higher throughput. Our analysis points out two cases where choosing minimal connectivity is

[1]When f and g are functions of N, we say that $f(N) = O(g(N))$ if there exists a scalar C such that $|f(N)| < Cg(N)$ for all N.

not optimal. In minimally connected Manhattan networks, every node has four neighbors. If nodes increase their transmission radius and gets connected also with neighbors of neighbors, we call it a *2-hop connected* Manhattan network. In Manhattan networks with nodes capable of receiving eight packets simultaneously, increasing connectivity from minimal to 2-hop yields about 54% increase in capacity. Simultaneous receptions is particularly relevant to networks with spread spectrum and multiple antennas; in such systems, we expect performance gains from non-minimal connectivity. In ring networks (Figure 1.c) the capacity is doubled by relaxing the minimal connectivity assumption. The optimal connectivity in ring goes to infinity as the network size grows. This result is true even without multipacket receptions. These examples show that minimal connectivity is not always optimal, and there are potential benefits of adaptive connectivity depending on the topology and channel usage.

1.3 Related Work

In the literature, the wireless network stability problems have been studied extensively both for networks with centralized scheduling [5]–[12] and the ALOHA protocol [13]–[18]. Our problem formulation is closest to the model used by Tassiulas and Ephremides in [5] where they studied the network stability with a specific probabilistic model and characterized the network stability region. They also gave an elegant throughput optimal policy that stabilizes the network at all arrival rates in the stability region. In [5], the packet arrival process is assumed to be independent and identically distributed, and the network stability is analyzed in a Markovian framework using the Lyapunov functional approach. In this paper, we consider more general (stationary and ergodic) arrival processes, and our stability notion is slightly different. We also use a different methodology (a dominant system approach) in stability analysis.

The network capacity problems have been studied in several contexts. The early works focused on the computation of achievable rates with distributed protocols such as ALOHA (*e.g.,* [19], [20], [4, p. 346]) and TDMA (*e.g.,* [21], [22]). Silvester and Kleinrock analyzed the capacities of regular networks with the slotted ALOHA protocol in [19], [20]. Using the collision channel they obtained the throughput of slotted ALOHA in regular networks. They showed that the minimal connectivity is optimal in Manhattan networks with slotted ALOHA, but in ring it is not. Later, Tsybakov and Bakirov studied the stability of multi-hop ALOHA networks [13]. Besides verifying some of the results in [19] from the stability point of view, Tsybakov and Bakirov obtained other general stability conditions for arbitrary networks. As outlined previously, our analysis for regular networks extend Silvester and Kleinrock's results in several directions considering centralized control as well as slotted ALOHA.

Gupta and Kumar [1] initiated a formal capacity analysis of random and arbitrary networks. Unlike most of the prior studies which started with a graph model having transmission powers fixed, Gupta and Kumar considered a joint optimization of transmission powers and schedules. They showed the fundamental result that the maximum per-node throughput scales proportional to $1/\sqrt{N}$. Our setup is different from that of [1] in that the network is *ergodic*; specifically, the topology and channel qualities form an ergodic process. In the ergodic network the node connections change in time according to certain statistics, and the links are not permanent as in [1]. For the regular networks in this paper we provide the capacity coefficients besides the scaling law. These coefficients, which are not apparent in [1], reveal considerable insights about the design of MAC and optimal node connectivity.

More recently, a number of other works [23]–[34] studied the capacity of wireless networks from various viewpoints. Toumpis and Goldsmith [25], [26] modelled the communication channel using deterministic rate matrices, and defined the notion of capacity region. They also analyzed the capacity regions of networks considering adaptive modulation and rates depending on the channel and interference conditions. Different from Toumpis and Goldsmith, we also consider randomness in receptions. We however do not consider rate adaptation.

The organization of the paper is as follows. In the next section the network model is introduced. In Section 3, the RTD policies are introduced, and the network stability and capacity regions are characterized. Also, an upper bound on the achievable rates is developed using the transport capacity. In Sections 4 and 5, proofs of the main theorems are given. In Section 6 we apply the developed tools to compute the capacity of regular networks. We conclude in Section 7.

Sets will be denoted by script letters. For a set $\mathcal{A}$, $|\mathcal{A}|$ is the number of elements in $\mathcal{A}$, and $\mathcal{A}^n = \{(a_1, a_2, \cdots, a_n) : a_i \in \mathcal{A}, i = 1, 2, \cdots, n\}$. The set of non-negative integers is $\mathcal{Z}_+ = \{0, 1, 2, \cdots\}$. For real valued vectors $A = (a_1, a_2, \cdots, a_n), B = (b_1, b_2, \cdots, b_n)$, we say that $A \leq B$ if $a_i \leq b_i$, for all i.

2 Network Model

Suppose that time is divided into unit length slots, and slot $t \in \mathcal{Z}_+$ is defined as the half-open interval $[t, t+1)$. Let $\mathcal{N} = \{1, 2, \cdots, N\}$ be the set of nodes in the network, and $\mathcal{L} = \{(i, j) : i, j \in \mathcal{N}, i \neq j\}$ be the set of links. For link $l = (i, j)$, the notation $t(l)$ denotes the transmitter node i, and $r(l)$ denotes the receiver node j.

There are multiple *traffic classes* in the network. Let $\mathcal{C}$ be the set of traffic classes. A packet from class $j \in \mathcal{C}$ will be called a j-packet. The destination of each j-packet is a single node $q(j)$. In this paper, at different places, we will use three ways of traffic classification. If the packets are classified according to their destinations, then $\mathcal{C} = \mathcal{N}$. The source and the destination of packets can be used for classification, in this case $\mathcal{C} = \{(i, j) : i, j \in \mathcal{N}, i \neq j\}$. Similarly, the packets can be classified according to the routes they follow, this will be discussed further in Section 3.1.

We will represent transmissions and the traffic classes of the transmitted packets using binary vectors. Let $E_{lj}(t)$ be equal to 1 if a j-packet is *transmitted* over link l in slot t, and 0 otherwise. Similarly, let $F_{lj}(t)$ be equal to 1 if a j-packet is *successfully received* over link l in slot t, and 0 otherwise. Over each link a single packet can be transmitted, that is, $\sum_j E_{lj}(t) \in \{0, 1\}$. Define $E(t) = (E_{lj}(t) : l \in \mathcal{L}, j \in \mathcal{C})$, and $F(t) = (F_{lj}(t) : l \in \mathcal{L}, j \in \mathcal{C})$. The set of transmissions in slot t is $\mathcal{E}(t) = \{l \in \mathcal{L} : E_{lj}(t) = 1 \text{ for some } j \in \mathcal{C}\}$. Similarly, define the set of receptions as $\mathcal{F}(t) = \{l \in \mathcal{L} : F_{lj}(t) = 1 \text{ for some } j \in \mathcal{C}\}$.

Time variation in the network topology and the channel qualities are modelled using *states*. Let $\mathcal{V}$ be the set of states, and $v(t) \in \mathcal{V}$ be the state of the network in slot t. The state $v(t)$ can be any network parameter affecting the receptions; examples include channel gains between users and spatial locations of the nodes. It is assumed that the process $(v(t) : t \in \mathcal{Z}_+)$ is stationary and ergodic, and the probability of state v is $p(v)$ in the stationary distribution[2].

[2]In the sequel, it is assumed that $\mathcal{V}$ is a countable set. However, this assumption is for notational convenience. Our results hold also when $\mathcal{V}$ is uncountable.

Wireless channels in general are subject to random fading, and neighboring transmissions interfere with each other. Because of these reasons, some of the transmitted packets may not be received successfully. We model the channel characteristics and the reception errors using a conditional probability density function (pdf) π. In each slot the received packets $\mathcal{F}(t)$ are determined randomly according to the pdf $\pi(\,\cdot\,; \mathcal{E}(t), v(t))$. The quantity $\pi(\mathcal{F}(t); \mathcal{E}(t), v(t))$ is the probability that the successful receptions are $\mathcal{F}(t)$ given that the transmissions are $\mathcal{E}(t)$ and the network is in state $v(t)$. The pdf π also specifies the *transmission constraints* (such as half-duplex nodes) and the *network topology*: if a set of transmissions $\mathcal{E}$ is physically impossible, then the set of successful receptions is empty with probability 1. In multi-hop networks there may be such impossible $\mathcal{E}$, since the nodes typically are restricted to communicate with neighbors. Specific choices of the pdf π give several previous models such as the collision channel [21], [22], the MPR model [2], [3], [35]–[40], and others [5]–[8], [41].

We assume that new packets arrive at the network randomly according to a stochastic process. Let $A_{ij}(t)$ be the number of j-packets arrived at node i in slot t. (Equivalently, we say that node i *generated* $A_{ij}(t)$ j-packets in slot t.) The arrival process $A(t) = (A_{ij}(t) : i \in \mathcal{N}, j \in \mathcal{C})$ is assumed to be stationary and ergodic[3] with mean $\lambda = (\lambda_{ij} : i \in \mathcal{N}, j \in \mathcal{C})$. In slot 0 the network starts operation with empty queues. The nodes can store an unlimited number of packets in their buffers, and a packet does not leave the network unless it reaches its destination. At time t the number of j-packets at node i is denoted by $n_{ij}(t)$. Define $n(t) = (n_{ij}(t) : i \in \mathcal{N}, j \in \mathcal{C})$ and $n_i(t) = \sum_j n_{ij}(t)$. Time evolution of each queue is described by

$$n_{ij}(t+1) = \begin{cases} n_{ij}(t) - \sum_{l \in \mathcal{L}:\ t(l)=i} F_{lj}(t) & \\ + \sum_{l \in \mathcal{L}:\ r(l)=i} F_{lj}(t) + A_{ij}(t), & \text{if } i \neq q(j) \\ & \\ 0, & \text{if } i = q(j). \end{cases} \tag{1}$$

In each slot t, a control policy determines $E(t)$. The policies we consider are causal, and they can be randomized. In slot t the policies assume the knowledge of $v(t)$.[4]

We have the following assumption which is expected to be satisfied in practice.

(A1) Define the marginal probability of successfully receiving the set $\mathcal{F}$ given $\mathcal{E}$ as

$$\Pi(\mathcal{F}; \mathcal{E}, v) = \sum_{\mathcal{F}':\ \mathcal{F} \subset \mathcal{F}'} \pi(\mathcal{F}'; \mathcal{E}, v).$$

If $\mathcal{E}'$ is another set of transmissions including $\mathcal{E}$, we require

$$\Pi(\mathcal{F}; \mathcal{E}, v) \geq \Pi(\mathcal{F}; \mathcal{E}', v), \tag{2}$$

that is the marginal probability of success is lower when there are more transmissions.

[3]The network state and the packet arrival processes are assumed to be *jointly* stationary and ergodic.

[4]The formal definition of a policy is the following. The information available up to time t is $I(t) = (A(r), F(r), v(r) : r = 0, 1, \cdots, t-1)$. Policies can be randomized; suppose that $U = (U(t) : t \in \mathcal{Z}_+)$ is a vector containing i.i.d. uniform [0,1] random variables which are used for randomization. A *policy* is a sequence of functions $\phi = (\phi_t : t \in \mathcal{Z}_+)$ such that $\phi_t(I(t), v(t), U) = E(t)$.

3 Main Results

In this section, we will characterize the capacity and stability regions of networks. In our characterization the so-called RTD policies play an important role. The basic idea behind the RTD policies is the assignment of random routes and the use of random schedules according to some probability distribution. In order to make these ideas precise we need a few more definitions that are presented in the next subsection.

3.1 Randomized Time-Division (RTD) Policies

A *path* from node i_0 to i_k is a vector $(i_0, i_1, \cdots, i_k) \in \mathcal{N}^{k+1}$ such that $i_0, \cdots, i_k$ are different nodes. Denote the set of all paths from i to j by $\mathscr{P}_{ij}$, and define $\mathscr{P} = \cup_{i,j \in \mathcal{N}} \mathscr{P}_{ij}$. For some $P = (i_0, i_1, \cdots, i_k) \in \mathscr{P}$, we say that link l is in path P (*i.e.,* $l \in P$) if $(t(l), r(l)) = (i_j, i_{j+1})$ for some $j \in \{0, \cdots, k-1\}$. Let $\mathscr{E}$ denote the power set of the set of links $\mathcal{L}$. A *routing vector* is a vector $H = (x_P \geq 0 : P \in \mathscr{P})$ satisfying

$$\sum_{P \in \mathscr{P}_{ij}} x_P = 1, \text{ for all } i \neq j.$$

Similarly, a *scheduling vector* is a vector $G = (p(\mathcal{E}; v) \geq 0 : v \in \mathcal{V}, \mathcal{E} \in \mathscr{E})$ satisfying

$$\sum_{\mathcal{E} \in \mathscr{E}} p(\mathcal{E}; v) = 1, \text{ for all } v \in \mathcal{V}.$$

An RTD policy is specified by the vectors G, H and arrival rate λ. The vectors G and H will be viewed as probability densities over routes and transmission schedules. Three mechanisms used in an RTD policy are the following.

Routing: The packets are identified with their routes, and every packet is assigned a fixed route randomly once it is generated. If node i generates a j-packet, route $P \in \mathscr{P}_{ij}$ is assigned with probability x_P.

Medium Access: In every slot, a randomly chosen schedule is applied. In slot t, transmission schedule $\mathcal{E} = \mathcal{E}(t)$ is chosen with probability $p(\mathcal{E}; v(t))$.

Queuing discipline: After the transmission schedule is chosen, every node chooses the types of packets it will transmit: If node i is scheduled to transmit over link l, a route P (such that $l \in P$) is chosen randomly with probability

$$Q(P, l) = \frac{x_P \lambda_{ij}}{\sum_{i,j \in \mathcal{N}} \sum_{P \in \mathscr{P}_{ij}:\ l \in P} x_P \lambda_{ij}}. \tag{3}$$

Then, over link l a packet with route P is transmitted if node i has a packet with route P. Equation (3) assures fairness: each route passing through link l is allocated bandwidth proportional to its traffic rate $x_P \lambda_{ij}$.

Now specification of the RTD policies is complete. Before analyzing network stability, we will discuss some connections between RTD policies and TDMA. In ad hoc networks TDMA can be done by applying a sequence of transmission schedules periodically (see [21], [22]). The knowledge of queue lengths is not required in TDMA, therefore it can be applied in a distributed network, but the cycles should be designed with a prior knowledge of arrival rates network should support. The medium access in RTD policies can be viewed as a generalization of TDMA to networks with time-variation. This generalization is done by choosing a random transmission schedule in each slot instead of cycling through different transmission schedules.

The RTD policies can be applied in distributed networks if every node has access to the network state $v(t)$ in each slot (This is the case if the network has a single state, or cycles through states periodically, or if there is a feedback link from a central controller broadcasting the state). One possibility for distributed implementation is via the use of pseudo-random number generators which were previously proposed in [42], [43]. In case all nodes use a common pseudo-randomization algorithm (or, a common seed [42]) then a pseudo-random vector $\mathcal{E}(t)$ can be picked according to distribution $p(\,\cdot\,; v(t))$ by each node locally. Once MAC is done using pseudo-randomization, routing and queuing discipline can be readily applied distributively.

3.2 Stability

In this subsection, we will define the stability and network flows. We will then characterize the network stability region using network flows.

Definition In a network $\mathcal{N}$ with arrival rate λ and with some policy, node $i \in \mathcal{N}$ is called *stable* if the distribution of queue length $n_i(t)$ converges to some proper distribution W as $t \to \infty$, that is

$$\lim_{t\to\infty} Pr\{n_i(t) < \theta\} = W(\theta) \quad \text{and} \quad \lim_{\theta\to\infty} W(\theta) = 1. \tag{4}$$

Node $i \in \mathcal{N}$ is called *substable* if

$$\lim_{\theta\to\infty} \liminf_{t\to\infty} Pr\{n_i(t) < \theta\} = 1. \tag{5}$$

We call a network is stable if all nodes in the network are substable; it is called unstable otherwise.

Substability is a condition weaker than stability: a stable node is always substable, but the converse is not always true. Substability admits a heuristic interpretation. Supposing that θ is the buffer capacity of node i, we can interpret $\limsup_{t\to\infty} Pr\{n_i(t) > \theta\}$ as the asymptotic buffer overflow probability of node i. A node is substable if and only if its asymptotic buffer overflow probability goes to zero as the buffer size θ tends to infinity.

Condition (5) is called substability by Loynes [44], and tightness by Billingsley [45]. In the wireless networking context, as a network stability criterion, substability is first used by Tsybakov and Bakirov [13]. Depending on the network model, other stability notions are also used in the literature [14]–[18], [5]–[12].

In network stability considerations we classify the packets according to their destinations, that is, $\mathcal{C} = \mathcal{N}$. Arrival rate $\lambda = (\lambda_{ij} : i, j \in \mathcal{N})$ is called *stabilizable* if there exists a policy that makes the network stable. The *stability region* of a network is the closure of the set of all stabilizable rates.

In order to characterize the stability region, we need to introduce the notion of feasible flows. Recall that $\Pi(\mathcal{F}; \mathcal{E}, v)$ is the marginal probability of success for set $\mathcal{F}$ given $\mathcal{E}$ is transmitted. In the following, to denote the marginal probability of success over link l we will use the notation $\Pi(l; \mathcal{E}, v)$ instead of $\Pi(\{l\}; \mathcal{E}, v)$.

Definition Rate $\lambda = (\lambda_{ij} \geq 0 : i, j \in \mathcal{N})$ is called *feasible* if there exist a scheduling vector $G = (p(\mathcal{E}; v) \geq 0 : v \in \mathcal{V}, \mathcal{E} \in \mathscr{E})$ and a routing vector $H = (x_P : P \in \mathscr{P})$ such that

$$\sum_{i,j \in \mathcal{N}} \sum_{P \in \mathscr{P}_{ij}:\ l \in P} x_P \lambda_{ij} \leq \sum_{v \in \mathcal{V}, \mathcal{E} \in \mathscr{E}} \Pi(l; \mathcal{E}, v) p(\mathcal{E}; v) p(v) \tag{6}$$

holds for all $l \in \mathcal{L}$. The *flow region* is the closure of the set of all feasible rates.

Our motivation for defining feasibility is the following. Consider an RTD policy with G and H. When the arrival rate is λ and the routing is done according to H, the traffic load on link $l \in \mathcal{L}$ is

$$\sum_{i,j \in \mathcal{N}} \sum_{P \in \mathscr{P}_{ij}:\ l \in P} x_P \lambda_{ij}.$$

The expected number of successful transmissions on link l is

$$\sum_{v \in \mathcal{V}, \mathcal{E} \in \mathscr{E}} \Pi(l; \mathcal{E}, v) p(\mathcal{E}; v) p(v) \tag{7}$$

given that $p(\mathcal{E}; v)$ is the fraction of time slots the transmission set $\mathcal{E}$ is used when the network is in state v. The RTD policy chooses each schedule $\mathcal{E}$ with probability $p(\mathcal{E}; v)$, but sometimes some other schedule $\mathcal{E}' \subset \mathcal{E}$ may be applied since nodes may run out of packets to transmit. In such cases, due to assumption (A1), the success probability over any link $l \in \mathcal{E}'$ does not decrease, and (7) can be thought as a worst case success rate over link l. Equation (6) ensures that the traffic load over each link is less than its worst case success rate. The following lemma asserts that this intuitive condition is sufficient for network stability.

Lemma 1 *If λ is feasible with scheduling vector G and routing vector H, then the RTD policy specified by G, H, λ stabilizes the network with arrival rate $(1-\epsilon)\lambda$ for all $\epsilon > 0$.*

Proof See Section 4. ∎

Next theorem can be considered as a converse to Lemma 1.

Theorem 1 *The stability and flow regions are identical.*

Proof Lemma 1 shows that the rates inside the flow region are stabilizable. For the converse, see Section 5. ∎

From the definition of flow region in (6), we note that it is only the *marginal* link success probabilities $\Pi(l; \mathcal{E}, v)$ —not the joint probabilities— that determine the flow region and, therefore by Theorem 1, the stability region.

Theorem 1 suggests a way to think about stability problems. To check if rate λ is stabilizable, we need to find a distribution G over schedules, and then we need to route packets according to another distribution H such that the traffic over each link is less than its success rate, that is, the rate λ is feasible with G and H. This approach is essentially similar to the standard flow approach (*e.g.,* Ford and Fulkerson [46]) that assigns a fixed capacity to each network link, and routes as much flow as possible from the source nodes to their destinations without violating

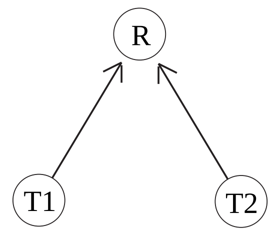

Figure 2: An example up-link network

the capacity of any link. In our network the link capacities are determined by the link success rates of a scheduling vector G. Once G is fixed, the standard flow approach [46] can be applied to obtain rates achievable with the particular G. Characterization of stability region using flows is typical in many other wired and wireless stochastic network models (*e.g.,* [5]–[12]).

It is interesting to notice that the Theorem 1 crucially depends on the assumption (A1). In general, it is true that the stability region is inside the flow region, but in networks violating (A1) feasible rates may not be stabilizable. For instance, consider the network depicted in Figure 2. Nodes T1 and T2 want to transmit packets to node R. The network has a single state, and there is a single class of traffic which is intended for node R. The channel reception probabilities is such that if the transmission set is $\mathcal{E} = \{(T1, R), (T2, R)\}$, then the reception is $\mathcal{F} = \{(T1, R)\}$ with probability 1. If $\mathcal{E}$ is either $\phi, \{(T1, R)\}$ or $\{(T2, R)\}$, then $\mathcal{F} = \phi$ with probability 1. In this network, the packets of $T1$ are successfully received only if $T2$ transmits at the same time, but in any case $T2$'s packets are not successfully received. The flow region can be obtained as $\{(\lambda_1, 0) : 0 \leq \lambda_1 \leq 1\}$ (The i'th entry shows the rate achievable by the i'th transmitter). However, the stability region is the set $\{(0, 0)\}$. To see this, observe that if the arrival rate for $T1$ is positive then the arrival rate of $T2$ should also be positive; this is so, since stabilization of the queue $T1$ requires node $T2$ to transmit simultaneously. However, if $T2$ has positive arrival rate it goes unstable whatsoever. Hence, in any case, one of the queues go unstable if either $T1$ or $T2$ has positive arrival rate. The assumption (A1) does not hold for this channel since the success probability of $T1$ increases when $T2$ transmits simultaneously. In this example we see that some additional conditions on the channel π are required in order to have the stability and the flow regions the same.

Some familiar results about slotted ALOHA, [18, Thm. 1] and, in part, [13, Thm. 4], can be obtained as special cases of Lemma 1. This is because the slotted ALOHA protocol (as described in [4, p. 348]) can be viewed as a special RTD policy. In a slotted ALOHA network, in case all nodes are backlogged, every node flips a coin and chooses to transmit or to listen with a fixed probability. If a node decides to transmit, the neighbor and the type of packet to transmit can be picked randomly according to a fixed probability distribution. Such a mechanism is a special case of assigning a probability distribution over the set of schedules *i.e.,* assigning a scheduling vector G. With this special structure on G, Lemma 1 gives an inner bound on the stability region of the slotted ALOHA, and [18, Thm. 1] and, in part [13, Thm. 4], are obtained as special cases.

3.3 Capacity

In this subsection, we will define the achievability of packet delivery rates and the capacity region. We will then argue that the capacity and stability regions are the same in networks satisfying assumption (A1).

Suppose that each packet is identified with its source and destination, *i.e.,* the set of traffic classes is $\mathcal{C} = \{(i,j) : i,j \in \mathcal{N}, i \neq j\}$. Let $s(k)$ be the source node, and $q(k)$ be the destination node for traffic class $k \in \mathcal{C}$. In the capacity problem the arrivals are not random, that is, $\lambda_{ik} = 0$ for all $i \in \mathcal{N}, k \in \mathcal{C}$. Nonetheless, the network starts with infinite number of packets waiting delivery at the source nodes, that is, $n_{ik}(0) = \infty$ for all $i = s(k)$. Let $W_{ij}(t)$ be the number of packets from class (i,j) successfully received by node j in slot t.

The notions of achievability (see [1]) and the capacity region are defined next.

Definition Rate $\lambda = (\lambda_{ij} \geq 0 : i,j \in \mathcal{N})$ is called *achievable* if there exists a network policy such that the average delivery rate is greater than λ, that is,

$$\liminf_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} W_{ij}(t) \geq \lambda_{ij} \quad \text{for all } i,j \in \mathcal{N} \tag{8}$$

is satisfied with probability one. The network *capacity region* is the closure of the set of achievable rates.

With this definition, we can state the following lemma analogous to the Lemma 1 of the previous subsection.

Lemma 2 *If λ is feasible with scheduling vector G and routing vector H, then the RTD policy*[5] *specified by G, H, λ achieves rate $(1-\epsilon)\lambda$ for all $\epsilon > 0$.*

Proof We prove this lemma in parallel with Lemma 1. Both proofs are given in Section 4. ■

Theorem 2 *The capacity and the flow regions are identical.*

Proof Lemma 2 shows that the flow region is inside the capacity region. The converse is proved in Appendix F. ■

Theorem 2 shows that the capacity region is completely specified by the marginal link success probabilities $\Pi(l;\mathcal{E},v)$ in (6).[6]

Surprisingly, Theorem 2 is valid even without assumption (A1). Let us motivate this with the example in the previous subsection. We have observed that $T1$ and $T2$ should transmit together to achieve non-zero rates. However, $T1$'s packets never get through and its buffer goes unstable when it has non-zero arrival rates. In the capacity problem, $T1$ already has infinitely many packets, and stability is not an issue. Therefore, every rate $(\lambda_1, 0)$, $0 \leq \lambda_1 \leq 1$, can be achieved if $T1$ and $T2$ transmit together in λ_1 fraction of the slots. This implies that the capacity region is $\{(\lambda_1, 0) : 0 \leq \lambda_1 \leq 1\}$ which is also the flow region. As in this example, Lemma 2 holds in networks without (A1), and the capacity and the flow regions are always the same. We further discuss why this result is in general true in Section 4.

[5] In a network applying RTD without random arrivals, we require the source nodes to regulate their traffic entering the network. This condition is needed to make the proof easier, and it is discussed in Section 4.2

[6] This is similar to the Shannon capacity of broadcast channels [47] that depends only on the marginal reception probabilities.

3.4 An Upper Bound Using Transport Capacity

In this subsection, we will introduce the notion of transport capacity, and develop an upper bound on achievable rates using the transport capacity. This upper bound is particularly useful in large networks where the exact computation of capacity region may not be computationally feasible.

Most of the wireless networks come with a notion of distance metric telling how close two nodes are. Some commonly used metrics are the Euclidean distance and the minimum number of hops required to reach from one node to another. Let $d(i,j)$ be the distance between nodes i and j. The distance metric $d(i,j)$ is assumed to satisfy the *triangle inequality* that is, for all $P \in \mathscr{P}_{ij}$,

$$d(i,j) \leq \sum_{l \in P} d(l),$$

where we use the notation $d(l)$ as a shorthand for $d(t(l), r(l))$. The usual definition of metric puts additional constraints of non-negativity and symmetry of $d(i,j)$ (see Rudin [48]). These constraints are not needed for the results in this paper.

The next proposition gives a necessary condition for achievability.

Proposition 1 *Let rate λ be in the capacity (or equivalently, in the stability or flow) region. Then,*

$$\sum_{i,j \in \mathcal{N}} \lambda_{ij} d(i,j) \leq \sum_{v \in \mathcal{V}} p(v) \sum_{l \in \mathcal{L}} d(l) \Pi(l; \mathcal{E}_v, v), \tag{9}$$

where

$$\mathcal{E}_v = \arg\max_{\mathcal{E} \in \mathscr{E}} \sum_{l \in \mathcal{L}} d(l) \Pi(l; \mathcal{E}, v).$$

Proof Let λ be feasible with a scheduling G and a routing vector H. Then,

$$\begin{aligned}
\sum_{i,j \in \mathcal{N}} \lambda_{ij} d(i,j) &\leq \sum_{i,j \in \mathcal{N}} \sum_{P \in \mathscr{P}_{ij}} \sum_{l \in \mathcal{L}: l \in P} x_P \lambda_{ij} d(l) && (10) \\
&= \sum_{l \in \mathcal{L}} d(l) \sum_{i,j \in \mathcal{N}} \sum_{P \in \mathscr{P}_{ij}: l \in P} x_P \lambda_{ij} \\
&\leq \sum_{l \in \mathcal{L}} d(l) \sum_{v \in \mathcal{V}, \mathcal{E} \in \mathscr{E}} \Pi(l; \mathcal{E}, v) p(\mathcal{E}; v) p(v) && (11) \\
&\leq \sum_{v \in \mathcal{V}} p(v) \sum_{l \in \mathcal{L}} d(l) \Pi(l; \mathcal{E}_v, v),
\end{aligned}$$

where (10) follows from the triangle inequality and (11) holds since λ is feasible. ∎

An interpretation of Proposition 1 is as follows. The quantity $\sum_{i,j \in \mathcal{N}} \lambda_{ij} d(i,j)$ can be viewed as the amount of work that needs to be done by carrying packets with rate λ. Similarly, $\sum_{l \in \mathcal{L}} d(l) \Pi(l; \mathcal{E}, v)$ is the expected progress, or work done, by using schedule $\mathcal{E} \in \mathscr{E}$. We call the right hand side of (9)

$$\sum_{v \in \mathcal{V}} p(v) \sum_{l \in \mathcal{L}} d(l) \Pi(l; \mathcal{E}_v, v) \tag{12}$$

as the *transport capacity* [1], which is the expected progress averaged over v, maximized with respect to $\mathcal{E}$ for each v. Informally speaking, Proposition 1 says that the total work that can be done by the network is always less than its transport capacity.

Definition Rate $\lambda' > 0$ is called *uniformly-achievable* if $(\lambda' 1(i \neq j) : i, j \in \mathcal{N})$ is in the capacity region, where $1(\cdot)$ is the indicator function. The *network capacity* η is $N-1$ times the maximum of the uniformly-achievable rates.

Multiplication by $N-1$ gives the sum rate delivered from any node to the other $N-1$ nodes, *i.e.,* per-node throughput. This notion of network capacity (with a different scaling) was previously used by Toumpis and Goldsmith [25]. The following theorem readily follows from the previous proposition and the definition of η.

Theorem 3 *An upper bound on network capacity* η *is given by*

$$\eta \leq \frac{1}{\bar{L}N} \sum_{v \in \mathcal{V}} p(v) \sum_{l \in \mathcal{L}} d(l) \Pi(l; \mathcal{E}_v, v), \tag{13}$$

where $\bar{L}$ *is the average distance between two arbitrarily selected nodes, i.e.,*

$$\bar{L} = \frac{1}{N(N-1)} \sum_{i,j \in \mathcal{N}} d(i,j), \tag{14}$$

and

$$\mathcal{E}_v = \arg\max_{\mathcal{E} \in \mathscr{E}} \sum_{l \in \mathcal{L}} d(l) \Pi(l; \mathcal{E}, v). \tag{15}$$

Proof Substitute $\lambda_{ij} = \lambda'$ in (9). After rearranging (9), observe that $(N-1)\lambda'$ is less than the right hand side of equation (13) for all λ'. ■

Theorem 3 is the main tool we will use for upper bounding the capacities of networks in Section 6. Recall that in our original formulation of feasibility we need to check the existence of two things: A scheduling vector and a routing vector. Theorem 3 simplifies our job by eliminating the routing vectors from the formulation. Even though the upper bound provided by Theorem 3 is not achievable in general, we will see that it is achievable in regular networks. Furthermore, we will observe another advantage of Theorem 3 in regular networks: It holds not only for the Euclidean metric but also any metric satisfying the triangle inequality.

4 Stability and Capacity with RTD Policies

In this section we discuss main ideas behind the achievability and stability results with RTD policies, and prove Lemma 1 and Lemma 2. In proving these lemmas we use the so-called dominant system approach (*e.g.,* [13], [14], [16]–[18]). That is, we first analyze a *heavy loaded* network where all nodes have packets waiting all the time. The heavy load assumption decouples an interacting queues problem into a series of queues problem whose stability is established using Loynes's theory [44]. We then provide a *stochastic ordering* relation (*e.g.,* [49]) between the normal[7] network and the heavy loaded network showing that the stability and the achievability in the heavy loaded network implies the stability and the achievability in the normal network.

While these ideas are well known in analysis of stochastic networks, their application to specific situations involve intricate details. The more complicated, somewhat unexpected, part in our proof is the stochastic ordering relation (Lemma 3) where we need the assumption (A1). For this part, we construct a probability space (where both the normal network and the heavy loaded network lives) in which it is shown that the number of packets waiting in the heavy loaded network is more than the number waiting in the normal network with probability 1.

4.1 Lemma 1: Feasibility Implies Stability

Throughout this subsection we will consider the stability setting: the packet arrivals are random, and the nodes start operation with empty queues. We will analyze an RTD policy determined by scheduling vector G, routing vector H and $\lambda = (\lambda_{ij} : i, j \in \mathcal{N})$. A network with an RTD policy operates as if the packets are classified according to their routes[8], and without loss of generality we can concentrate on a network with $\mathcal{C} = \mathscr{P}$. We call a packet with route P as a P-packet.

In this section we will use the following notation for equation (1),

$$n_{i,P}(t+1) = \begin{cases} n_{i,P}(t) - \sum_{l\in\mathcal{L}:\ t(l)=i} F_{l,P}(t, E(t), v(t)) \\ + \sum_{l\in\mathcal{L}:\ r(l)=i} F_{l,P}(t, E(t), v(t)) + A_{i,P}(t), & \text{if } i \neq q(P) \\ 0, & \text{if } i = q(P). \end{cases} \tag{16}$$

That is, we replace the notation for $F_{l,P}(t)$ by $F_{l,P}(t, E(t), v(t))$ that also indicates the state $v(t)$ and the transmissions $E(t)$.

Let $D_{l,P}(t)$ be equal to 1 if the RTD policy has chosen to transmit a packet with route P over link l, and 0 otherwise. Recall that even though $D_{l,P}(t)$ is 1, a packet over link l is not transmitted if the scheduled transmitter node does not have any P-packets *i.e.*, $n_{t(l),P} = 0$. This definition helps us to express the operation of the RTD policy concisely:

$$E(t) = (D_{l,P}(t) 1(n_{t(l),P}(t) > 0) : l \in \mathcal{L}, P \in \mathcal{C}). \tag{17}$$

Now, we are at the position to describe the heavy loaded network. Define $D(t) = (D_{l,P}(t) : l \in \mathcal{L}, P \in \mathcal{C})$. The queue lengths $n^*_{i,P}(t)$ in a heavy loaded network evolve as follows

$$\begin{aligned} n^*_{i,P}(t+1) = n^*_{i,P}(t) + A_{i,P}(t) - &\sum_{l\in\mathcal{L}:\ t(l)=i} F_{l,P}(t, D(t), v(t))\ 1(n^*_{t(l),P}(t) > 0) \\ + \sum_{l\in\mathcal{L}:\ r(l)=i} &F_{l,P}(t, D(t), v(t))\ 1(n^*_{t(l),P}(t) > 0), \end{aligned} \tag{18}$$

[7] In this section we refer to the network we have been considering so far as the *normal* network to distinguish it from the heavy loaded one.

[8] There exists a subtle mathematical detail. An RTD policy assigns a random route to each arriving packet. Even in a network with $\mathcal{C} = \mathcal{N}$, the policy operates as if the packets are classified according to their routes. We can think of all networks with an RTD policy as $\mathcal{C} = \mathscr{P}$ such that the arrival rates scale accordingly, that is, if the arrival rate in the normal network is $(\lambda_{ij} : i, j \in \mathcal{C})$, then the arrival rate of packets with route $P \in \mathscr{P}_{ij}$ is $\lambda_{ij} x_P$.

if $i \neq q(P)$. To obtain (18) from (16), we moved the indicator function $1(\cdot)$ in equation (17) to outside of $F(\cdot)$ as a multiplicative factor. In the normal network the nodes which are scheduled to transmit may not transmit since they may not have a packet to transmit. In the heavy loaded network, on the contrary, the receptions $F(\cdot)$ are determined as if the set of transmissions is $D(t)$ in each slot, *i.e.*, $E(t) = D(t)$. We use the term "heavy loaded" because $E(t) = D(t)$ is possible only if every scheduled transmitter has packets waiting all the time, that is, the nodes are heavily loaded.

The following proposition asserts the stability of the heavy loaded network.

Proposition 2 *Let λ be feasible with G and H. For some $\epsilon > 0$, let the arrival rate of P-packets ($P \in \mathscr{P}_{ij}$) be*

$$\lambda_P = (1-\epsilon)\lambda_{ij} x_P. \tag{19}$$

Then, the nodes in the heavy loaded network are stable, i.e., for each i, P there exists $W(\cdot)$ such that

$$\lim_{t\to\infty} Pr\{n^*_{i,P}(t) < \theta\} = W(\theta) \quad \textit{and} \quad \lim_{\theta\to\infty} W(\theta) = 1. \tag{20}$$

Furthermore, $\frac{1}{t} n^*_{i,P}(t) \to 0$ almost surely as $t \to \infty$.

Proof In a network with an RTD policy, packets from each traffic class follow a series of queues. The stability of the network follows from a standard application of Loynes's theory [44] for series of queues. In the following, we will discuss the rationale behind the proposition. The details of how Loynes's theory applies are given in Appendix A.

Let l be a link in path P. The analysis of the heavy loaded network is much simpler than the normal network, because the event of successful transmission of P-packets over link l does not depend on queue lengths at other nodes. That is, regardless of what is happening at the other queues, the P-packets are successfully transmitted over link l according to the process $(F_{l,P}(t, D(t), v(t)) : t \in \mathcal{Z}_+)$, and it is the mean of this process that determines the stability.

From the definition of RTD policies, it follows that the mean of $F_{l,P}(\cdot)$ is

$$\mathbb{E}\{\ F_{l,P}(t, D(t), v(t))\ \} = Q(P,l) \sum_{v\in\mathcal{V}, \mathcal{E}\in\mathscr{E}} \Pi(l;\mathcal{E},v)p(\mathcal{E};v)p(v). \tag{21}$$

If $P \in \mathscr{P}_{ij}$, then because of feasibility (6) and the definition (3) of $Q(P,l)$,

$$\lambda_{ij} x_P \leq \mathbb{E}\{\ F_{l,P}(t, D(t), v(t))\ \}.$$

Moreover, since the arrival rate of P-packets λ_P is strictly smaller than $\lambda_{ij} x_P$, it follows that

$$\lambda_P < \mathbb{E}\{\ F_{l,P}(t, D(t), v(t))\ \}. \tag{22}$$

The above equation is a throughput condition: arrival rate for class P (left hand side) is strictly smaller than the expected number of P-packets transmitted over link l (right hand side) for each link $l \in P$. This, exactly, is the condition required in Loynes's theory to show the stability of the series of queues delivering P-packets.

The second statement in the proposition, $\frac{1}{t} n^*_{ij}(t) \to 0$ almost surely, follows from the convergence arguments given in Section 2.32 in [44]. For further details about the connection with Loynes [44], see Appendix A. ■

Next lemma gives the previously mentioned stochastic ordering relation.

Lemma 3 *For each $P \in \mathcal{C}$, the total number of P-packets in the heavy loaded network is stochastically larger than the total number P-packets in the normal network, i.e.,*

$$Pr(\sum_{i\in\mathcal{N}} n^*_{i,P}(t) > \theta) \;\geq\; Pr(\sum_{i\in\mathcal{N}} n_{i,P}(t) > \theta), \tag{23}$$

for all t and θ. Moreover, under the conditions in Proposition 2, $\frac{1}{t}n_{i,P}(t)\to 0$ almost surely as $t\to\infty$.

Proof See Appendix B. ∎

Now we can prove the stability of the network without heavy loaded transmissions. Under the conditions in Proposition 2, (20) implies that each $n^*_{i,P}(t)$ is a substable sequence. Sums of nonnegative substable sequences is substable (see Szpankowski [16]), therefore, $\sum_{i\in\mathcal{N}} n^*_{i,P}(t)$ is substable. From (23), this implies that $\sum_{i\in\mathcal{N}} n_{i,P}(t)$ is substable. Since a nonnegative sequence smaller than a substable sequence is substable, each $n_{i,P}(t)$ in the normal network is substable. Again using the fact that sum of substable sequences is substable, we see that each $n_i(t) = \sum_{P\in\mathscr{P}} n_{i,P}(t)$ is substable. Hence, the proof of Lemma 1 is complete.

4.2 Proof of Lemma 2

First, let's assume that the arrivals in the network are random as considered in the previous subsection. Lemma 3 shows that $\frac{1}{t}n_{i,P}(t)\to 0$, for all $i \in \mathcal{N}, P \in \mathcal{C}$. This implies that the delivery rate of P-packets is equal to the arrival rate which is $\lambda_P = (1-\epsilon)\lambda_{ij}x_P$. Therefore, the total delivery rate of packets from i to j is $\sum_{P\in\mathscr{P}_{ij}} \lambda_P = (1-\epsilon)\lambda_{ij}$.

In the capacity problem, every node has infinitely many packets waiting to be delivered, and the arrivals are not random. However, if the source nodes *regulate* the traffic incoming to the network and operate as if the arrivals are random, then the result in the previous paragraph is applicable. What we mean by regulation is that node i should introduce its P-packets into the network with rate λ_P according to a stationary and ergodic process. Given that the nodes operate in this way the rate $(1-\epsilon)\lambda$ is achieved using an RTD policy, and Lemma 2 follows.

4.3 Achievability of Flow Region Without Assumption (A1)

For the argument in the previous subsection we need the assumption (A1) that is used in Lemma 3. However, it is in general true that all rates inside the flow region are achievable. To see this, we will use a slightly modified form of RTD policies: if a P-packet chosen for transmission over link l (*i.e.,* $D_{l,P}(t) = 1$), but the transmitter node $t(l)$ doesn't have any P-packet, then let node $t(l)$ transmit another packet with source $t(l)$ and destination $r(l)$ over link l (there are infinitely many such packets in node $t(l)$'s buffer). Besides these extra packet transmissions, the network under this policy operates no different from a heavy loaded network; specifically, the reception statistics is determined according to the set of transmissions $\mathcal{D}(t) = \{l \in \mathcal{L} : D_{l,P}(t) = 1$ for some $P \in \mathcal{C}\}$, which is the same in the heavy loaded network. The results of Proposition 2 hold also for this network. Proposition 2 implies that delivery rate λ_P is achieved for each $P \in \mathcal{C}$, and therefore $(1-\epsilon)\lambda$ is achieved.

5 Theorem 1: Stability Implies Feasibility

In this section, we will prove the converse part of Theorem 1. Our proof is constructive. We will consider a stable network with arrival rate λ, and by using certain statistics of the network, we will construct a scheduling vector G and a routing vector H that make $\lambda - \epsilon 1_\lambda \geq 0$ feasible. The scalar ϵ is positive, and the notation 1_λ is a shorthand for $\big(1(\lambda_{t(l),r(l)} > 0) \ : \ l \in \mathcal{L}\big)$.

By adding up equations for $t = 1, 2, \cdots, T$ in (1), we see that

$$n_{ij}(T) = \sum_{t=0}^{T-1} \left[A_{ij}(t) + \sum_{l \in \mathcal{L}:\ r(l)=i} F_{lj}(t) - \sum_{l \in \mathcal{L}:\ t(l)=i} F_{lj}(t) \right] \tag{24}$$

holds for all $T \in \mathcal{Z}_+$, $i \neq j$. The following lemma relates stability with the expected queue length.

Lemma 4 *If the network is stable, then for all $i \in \mathcal{N}$,*

$$\frac{1}{t} \mathbb{E} n_i(t) \to 0 \quad as \quad t \to \infty. \tag{25}$$

Proof See Appendix C ■

Suppose that at time instant $T \in \mathcal{Z}_+$

$$\mathbb{E}\{\frac{1}{T} \sum_{i \in \mathcal{N}} n_i(T)\} < \epsilon \tag{26}$$

is satisfied. Existence of such a T is guaranteed by the previous Lemma. Define

$$p(\mathcal{E}; v) \triangleq \frac{1}{T} \sum_{t=0}^{T-1} \Pr\{\mathcal{E}(t) = \mathcal{E} | v(t) = v\}, \tag{27}$$

and the scheduling vector $G = (p(\mathcal{E}; v) : v \in \mathcal{V}, \mathcal{E} \in \mathscr{E})$.

Lemma 5 *The scheduling vector G defined by (27) satisfies*

$$\sum_{j \in \mathcal{N}} \mathbb{E}\{\frac{1}{T} \sum_{t=0}^{T-1} F_{lj}(t)\} = \sum_{v \in \mathcal{V}, \mathcal{E} \in \mathscr{E}} \Pi(l; \mathcal{E}, v) p(\mathcal{E}; v) p(v), \tag{28}$$

for all $l \in \mathcal{L}$.

Proof See Appendix D ■

Define

$$\begin{aligned} e_{lj} &= \mathbb{E}\{\frac{1}{T} \sum_{t=0}^{T-1} F_{lj}(t)\}, \\ \gamma_{ij} &= \mathbb{E}\{\frac{1}{T} n_{ij}(T)\}. \end{aligned}$$

The next proposition provides a routing vector H which will be used to show that rate $\lambda - \epsilon 1_\lambda$ is feasible.

Proposition 3 *Given* $\epsilon > 0$ *and the vectors* $(e_{lj} \geq 0 : l \in \mathcal{L}, j \in \mathcal{N})$, $\lambda = (\lambda_{ij} \geq 0 : i, j \in \mathcal{N})$, $\gamma = (\gamma_{ij} \geq 0 : i, j \in \mathcal{N})$ *such that for all* $i, j \in \mathcal{N},\ i \neq j$,

$$\textit{(i)} \quad \lambda_{ij} - \gamma_{ij} = \sum_{l \in \mathcal{L}:\ t(l)=i} e_{lj} - \sum_{l \in \mathcal{L}:\ r(l)=i} e_{lj}, \tag{29}$$

$$\textit{(ii)} \quad \lambda_{jj} = \gamma_{jj} = \sum_{l \in \mathcal{L}:\ t(l)=j} e_{lj} = 0, \tag{30}$$

$$\textit{(iii)} \quad \sum_{i,j \in \mathcal{N}} \gamma_{ij} < \epsilon, \tag{31}$$

$$\textit{(iv)} \quad \lambda - \epsilon 1_\lambda \geq 0, \tag{32}$$

are satisfied. Then, there exists a routing vector $H = (x_P \geq 0 : P \in \mathscr{P})$ *such that*

$$\sum_{i,j \in \mathcal{N}} \sum_{P \in \mathscr{P}_{ij}:\ l \in P} x_p(\lambda_{ij} - \epsilon 1(\lambda_{ij} > 0)) \leq \sum_{j \in \mathcal{N}} e_{lj} \tag{33}$$

holds for all $l \in \mathcal{L}$.

Proof See Appendix E. ■

Next, we will argue that $\lambda - \epsilon 1_\lambda$ is feasible. When we multiply both sides in (24) with $\frac{1}{T}$ and take the expectation, we see that the above condition 3(i) is satisfied. Condition 3(ii) holds as a result of our particular choices for γ and $(e_{lj} : l \in \mathcal{L}, j \in \mathcal{N})$. Condition 3(iii) holds because of (26). All conditions of Proposition 3 are satisfied, and we can apply it. Proposition 3 guarantees the existence of H satisfying (33). Lemma 5 gives

$$\sum_{j \in \mathcal{N}} e_{lj} = \sum_{v \in \mathcal{V}, \mathcal{E} \in \mathscr{E}} \Pi(l; \mathcal{E}, v) p(\mathcal{E}; v) p(v).$$

The previous equality together with (33) imply

$$\sum_{i,j \in \mathcal{N}} \sum_{P \in \mathscr{P}_{ij}:\ l \in P} x_P(\lambda_{ij} - \epsilon 1(\lambda_{ij} > 0)) \leq \sum_{v \in \mathcal{V}, E \in \mathscr{E}} \Pi(l; \mathcal{E}, v) p(\mathcal{E}; v) p(v), \tag{34}$$

for all $l \in \mathcal{L}$. That is, $\lambda - \epsilon 1_\lambda$ is feasible.

6 Applications

In this section, we compute the capacity of regular networks and provide capacity achieving RTD policies. We will first introduce the MPR model. We will then analyze the capacity of Manhattan networks in a variety of settings, and compute the capacity of ring networks in the end.

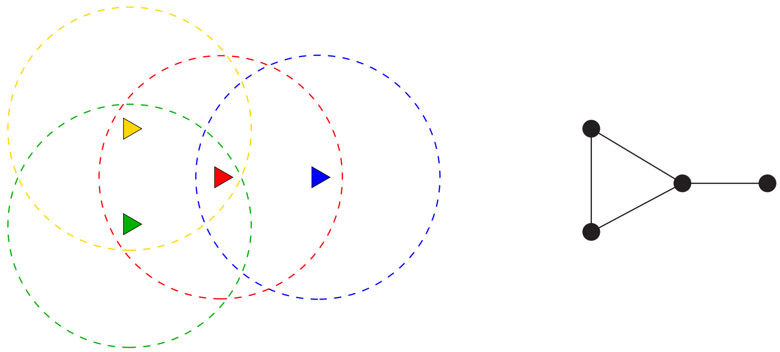

Figure 3: Nodes and transmission ranges in a planar network are shown on the left. The right figure shows the corresponding graph.

6.1 Multipacket Reception Channel

We consider networks represented with an undirected graph such that two nodes i and j can communicate directly only if they are connected with an edge. Classically, the graph models are used with the collision channel assumption [50]: Two nodes can communicate directly if they are within a distance r. Transmission from node i to node j is successful only if there is no other transmitter within distance r to node j (see Figure 3).

In wireless networks with CDMA (code-division multiple access) and/or multiple antennas, the collision channel assumptions do not hold, that is, the nodes might be capable of receiving multiple packets simultaneously, and there may be unexpected reception errors due to channel time-variation. To be able to consider such networks, we use the graph models with MPR [2], [3], [35]–[40]. Suppose that nodes can not transmit and receive at the same time. Each node can transmit at most one packet at a time. In each slot, a node can correctly receive and decode a fraction of the number of transmissions in its neighborhood. The reception probabilities are given by the *Receiver MPR Matrix* $\mathbf{C}$. The entries of the MPR matrix are given as

$$C_{n,k} = \Pr\{k \text{ packets are received} \mid n \text{ packets are transmitted in the neighborhood}\}.$$

The *Receiver MPR Matrix* $\mathbf{C}$ is defined by

$$\mathbf{C} = \begin{pmatrix} C_{1,0} & C_{1,1} & & \\ C_{2,0} & C_{2,1} & C_{2,2} & \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{35}$$

It is assumed that given the transmitting nodes in the network, the successful reception events of different receivers are independent.

Some examples of MPR are the *collision channel* $\mathbf{C}_1$ and the *2-collision channel* $\mathbf{C}_2$,

$$\mathbf{C}_1 = \begin{pmatrix} 0 & 1 & & & \\ 1 & 0 & 0 & & \\ 1 & 0 & 0 & 0 & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \mathbf{C}_2 = \begin{pmatrix} 0 & 1 & & & \\ 0 & 0 & 1 & & \\ 1 & 0 & 0 & 0 & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{36}$$

As a generalization of $\mathbf{C}_1$ and $\mathbf{C}_2$, we define the *M-collision channel* $\mathbf{C}_M$, [35]–[37], in which simultaneous reception of less than or equal to M packets is possible; if more then M packets are transmitted, then none of them are correctly received.

The MPR model is a special probabilistic model, and we need a condition equivalent to assumption (A1) in order to apply the results in Section 3. Let $\mathcal{E} = \{1, 2, \cdots, n\}$ be the set of transmitting nodes neighboring a receiver, $\{i_1, i_2, \cdots, i_k\}$ be any k-element subset of $\{1, 2, \cdots, n\}$, and $\mathcal{F}$ be the set of correctly received packets by the receiver. Note that the number of k-element subsets of $\mathcal{E}$ is $\binom{n}{k}$, and an assumption of the MPR model is that the reception event of every k-element subset is with equal probability. Then,

$$\Pr\{\mathcal{F} = \{i_1, i_2, \cdots, i_k\} \mid \mathcal{E} = \{1, 2, \cdots, n\}\} = \frac{C_{n,k}}{\binom{n}{k}}. \tag{37}$$

Define

$$\tilde{C}_{n,k} = \Pr\{\{i_1, i_2, \cdots, i_k\} \subset \mathcal{F} \mid \mathcal{E} = \{1, 2, \cdots, n\}\}, \tag{38}$$

which is the marginal probability of success for the transmitters $\{i_1, i_2, \cdots, i_k\}$. A simple counting argument shows that

$$\tilde{C}_{n,k} = \sum_{m=k}^{n} \binom{n-k}{m-k} \frac{C_{n,m}}{\binom{n}{m}}. \tag{39}$$

We require the marginal probability of success to be lower when there are more transmissions, *i.e.*,

$$\tilde{C}_{n_1,k} \geq \tilde{C}_{n_2,k} \tag{40}$$

for all $k \leq n_1 \leq n_2$. This condition is equivalent to assumption (A1), and it eliminates MPR matrices such as

$$\mathbf{C} = \begin{pmatrix} 1 & 0 & & \\ 0 & 0 & 1 & \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{41}$$

which is not encountered in practice.

6.2 Capacity of Manhattan Networks

In the previous sections we have established all the tools necessary for analyzing the capacity of regular networks. Before going into the details, we will first outline our methodology. Our main tool for upper bounding the capacities of networks is Theorem 3. In order to apply Theorem 3 we will compute the average path length $\bar{L}$ and the transport capacity. In most of the networks considered, the network has only one state. Hence, the computation of (13) requires the maximization in (15) only for a single state.

After finding an appropriate upper bound on network capacity, we will show that the upper bound is achieved exactly, or approximately with an error of the order $O(1/N)$ (or with an error $O(1/N^2)$ in ring networks). Specifically, we will find routing and scheduling vectors such that the corresponding RTD policy achieves the upper bound. The basic idea behind optimal routing in regular networks is to use shortest paths while balancing the routing load. On the other hand, we will see that the optimal MAC problem is equivalent to packing the maximum number of transmissions into a regular lattice.

A node in the Manhattan network is determined by two coordinates $(x, y) \in \{0, \cdots, \sqrt{N}-1\} \times \{0, \cdots, \sqrt{N}-1\}$. We define the distance between two nodes (x_0, y_0) and (x_1, y_1) as the minimum number of hops to reach from one node to another, that is,

$$d\{(x_0, y_0), (x_1, y_1)\} = \min\{\delta x, \sqrt{N} - \delta x\} + \min\{\delta y, \sqrt{N} - \delta y\}, \tag{42}$$

where $\delta x = |x_0 - x_1|$ and $\delta y = |y_0 - y_1|$. Recall that the nodes on one edge of the Manhattan network are connected to the nodes on the opposite edge; because of this property the distance metric is defined as (42) instead of $d\{(x_0, y_0), (x_1, y_1)\} = \delta x + \delta y$. It can be easily seen that $d\{\cdot\}$ satisfies the triangle inequality. A simple calculation yields the following proposition.

Proposition 4 *In the Manhattan network with N nodes, the average distance between two nodes $\bar{L}$ is given by $\sqrt{N}/2 + O(1/\sqrt{N})$, or more precisely,*

$$\bar{L} = \begin{cases} \frac{\sqrt{N}}{2}, & \sqrt{N} \text{ odd} \\ \frac{N\sqrt{N}}{2(N-1)}, & \sqrt{N} \text{ even} \end{cases} \tag{43}$$

Proof See Appendix G. ∎

The following lemma will be used to show the achievability of the capacity of Manhattan networks.

Lemma 6 *In the Manhattan network, there exists a routing vector $H = (x_P : P \in \mathscr{P})$ such that for every link l between two neighbors*

$$\sum_{i,j \in \mathcal{N}} \sum_{P \in \mathscr{P}_{ij}:\ l \in P} x_P = \frac{(N-1)\bar{L}}{4}. \tag{44}$$

Proof See Appendix H. ∎

Lemma 6 is a load balancing property. Equation (44) says that, in a network with unit arrival rates (*i.e.,* $\lambda_{ij} = 1, i \neq j$), there exists a routing vector uniformly distributing the traffic load over links. In the proof of Lemma 6, we have shown that every *symmetric*, *shortest path* routing vector satisfies (44). Since such a routing vector always uses shortest paths, the quantity in equation (44) is the minimum load that can be put over links uniformly.

Define

$$C_n = \sum_{k=1}^{n} k C_{n,k}$$

which is the expected number of correctly received packets given n packets are transmitted. Next theorem characterizes the capacity of Manhattan networks.

	$\tau = 1$	$\tau = 2$	$\tau = 3$	$\tau = 4$
e_τ	4	3	4	5

Table 1: e_τ versus τ.

Theorem 4 (The capacity of Manhattan networks) *Let η be the capacity of a Manhattan network of N nodes each with MPR matrix $\mathbf{C}$. Define*

$$\eta^* = \max_{i=1,\cdots,4} \frac{C_i}{i+1} \frac{1}{\bar{L}}. \tag{45}$$

The following relations hold

$$\begin{aligned} \eta &\leq \eta^* && (46) \\ \eta &= \eta^* + O(\frac{1}{N}). && (47) \end{aligned}$$

Furthermore, if $\sqrt{N}$ is divisible by e_τ, then $\eta = \eta^*$, where

$$\tau = \arg\max_{i=1,\cdots,4} \frac{C_i}{i+1}, \tag{48}$$

and e_τ is given in Table 1.

Proof First, we will argue that

$$\frac{1}{\bar{L}N} \sum_{l \in \mathcal{L}} \Pi(l; \mathcal{E}, v) \leq \eta^*, \tag{49}$$

for every transmission set $\mathcal{E}$. In the considered network there is a single state, the distance of every link $d(l)$ between neighboring nodes is 1, and the previous inequality proves $\eta \leq \eta^*$ as a result of Theorem 3.

To see (49), we will classify the nodes in the network according to the transmission set $\mathcal{E}$. Every node either transmit a packet or stay in the reception mode. Every node k in the reception mode receives two types of packets: the packets intended for the receiver k, and the packets intended for other nodes. Let $\mathcal{A}_{ij}$ be the set of nodes k such that k does not transmit, k receives j packets for itself, and k receives $i-j$ packets transmitted for other nodes (see Figure 4). Define $A_{ij} = |\mathcal{A}_{ij}|$ as the number nodes in $\mathcal{A}_{ij}$. Note that A_{ij} can be non-zero only for $0 \leq j \leq i \leq 4$, since nodes can receive packets from at most four other nodes. Every node in the network can transmit one packet at a time and for every receiver in set $\mathcal{A}_{ij}$ there exists j other transmitters in the network. The A_{ij} must satisfy

$$\sum_{i=0}^{4} \sum_{j=0}^{i} (1+j) A_{ij} \leq N, \tag{50}$$

since the total number of nodes in the network is N.

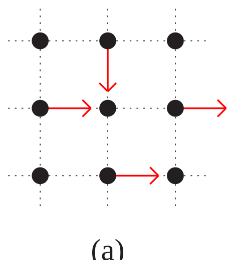

Figure 4: These figures illustrate the definition of $\mathcal{A}_{ij}$. In (a), the node in the center receives 2 packets intended for itself, and 2 packets intended for other nodes; therefore, it is an element of the set $\mathcal{A}_{4,2}$. The node in the center in (b) receives 2 packets for itself, and 1 packet intended for some other node; it is in $\mathcal{A}_{3,1}$. In (c) the node in the center is not an element of any $\mathcal{A}_{i,j}$ since it is transmitting.

For a node receiving j packets intended for itself and receiving $i-j$ packets intended for other nodes, the expected number of correctly received packets for itself is $\frac{j}{i}C_i$ (see Appendix I). Therefore, the expected number of successful transmissions divided by $\bar{L}N$ is

$$\frac{1}{\bar{L}N}\sum_{l\in\mathcal{L}}\Pi(l;\mathcal{E},v) = \frac{1}{\bar{L}N}\sum_{i=0}^{4}\sum_{j=0}^{i}\frac{j}{i}C_iA_{ij}. \tag{51}$$

Consider the optimization problem

$$\begin{aligned} \text{maximize} \qquad & \xi = \frac{1}{\bar{L}N}\sum_{i=0}^{4}\sum_{j=0}^{i}\frac{j}{i}C_iA_{ij} \\ \text{subject to} \qquad & \sum_{i=0}^{4}\sum_{j=0}^{i}(1+j)A_{ij} \le N \\ & A_{ij} \ge 0, \end{aligned} \tag{52}$$

where the maximization is with respect to *real* valued A_{ij}. In the original problem, the A_{ij} can only take integer values. Since we relax this constraint (and some others), the solution of the above optimization yields an upper bound on (51).

The equation (52) is a linear programming problem, and its solution is well known to be at one of the extreme points of the constraint set. Namely, the solution is attained at

$$A_{ij} = \begin{cases} \frac{N}{j'+1}, & \text{if } i=i',\ j=j' \\ 0, & \text{otherwise,} \end{cases} \tag{53}$$

for some $0 \le j' \le i' \le 4$. When we substitute the possible candidates for A_{ij} in (52), it is seen

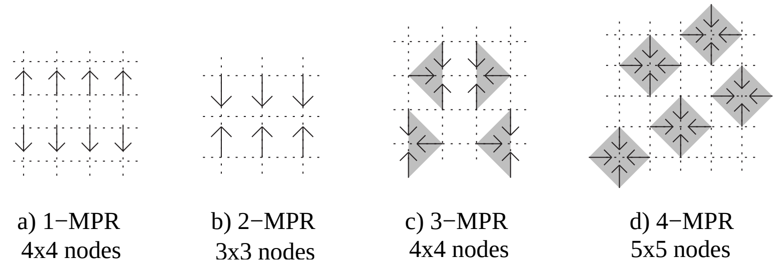

Figure 5: Different scheduling patterns for τ-MPR, $\tau \in \{1, 2, 3, 4\}$. Dashed lines show the links, the arrows show scheduled packet transmissions.

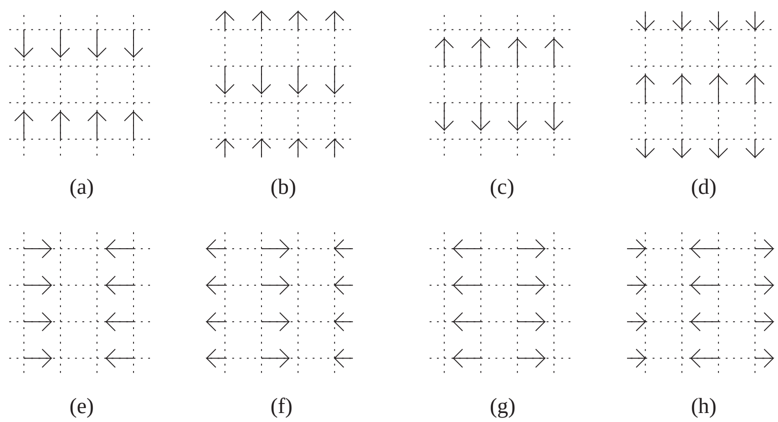

Figure 6: This is a network with 16 nodes. Eight phases of 1-MPR scheduling are shown. In each slot, the medium access protocol applies a randomly selected phase with 1/8 probability.

that

$$\begin{aligned} \xi &= \frac{1}{\bar{L}N} \sum_{i=0}^{4} \sum_{j=0}^{i} \frac{j}{i} C_i A_{ij} \\ &= \frac{1}{\bar{L}} \frac{j'}{i'(j'+1)} C_{i'} \\ &\leq \frac{1}{\bar{L}} \frac{C_{i'}}{i'+1} \\ &\leq \eta^*. \end{aligned} \tag{54}$$

The first inequality is due to the fact that $j'/(j'+1)$ is an increasing function of j', and j' is upper bounded by i'. Thus, (49) holds and η^* is an upper bound on the network capacity.

Next, we will show that $\eta^* + O(1/N)$ is achievable. For this, we will use the RTD policy specified by the routing vector in Lemma 6 and a special scheduling vector which will be called *τ-MPR scheduling*. The τ used to achieve $\eta^* + O(1/N)$ is defined in (48). In τ-MPR scheduling, the network is tiled using the τ-MPR pattern (see Figure 5) and its shifted/rotated versions. In τ-MPR, every scheduled receiver receives τ packets intended for itself. It can be observed that the τ-MPR patterns can tile the network if and only if $\sqrt{N}$ is divisible by e_τ.

To demonstrate the use of τ-MPR scheduling, suppose that $\tau = 1$ (that is, $\eta^* = C_1/2\bar{L}$) and $\sqrt{N}$ is divisible by 4. In this case, 1-MPR pattern and its shifted/rotated versions (Figure 6) can tile all of the network. We call each shifted/rotated version of 1-MPR pattern as a *phase* of 1-MPR scheduling. For medium access, we assign 1/8 probability to each phase in Figure 6. With this assignment of probabilities each node gets a chance to transmit to each neighbor with probability 1/8. The probability of success of each transmission is C_1. As a result of this, every node successfully transmits $C_1/8$ packets on the average to each of its neighbors. Supposing that 1-MPR scheduling is used together with the routing vector provided by lemma 6, all rates λ satisfying

$$\frac{(N-1)\lambda\bar{L}}{4} \leq \frac{C_1}{8} \tag{55}$$

are uniformly achievable. This shows that $\lambda = \eta^*/(N-1)$ is uniformly achievable.

Using arguments identical to the one above, it can observed that $\lambda = \eta^*/(N-1)\eta^*$ can be achieved uniformly using τ-MPR scheduling whenever $\sqrt{N}$ is divisible by e_τ. In case $\sqrt{N}$ is not divisible by e_τ, τ-MPR patterns can tile all of the network but a small portion. It can be observed that the number of nodes which can be scheduled with τ-MPR is $N + O(\sqrt{N})$. Again by using shifted/rotated versions of the τ-MPR, over each link

$$\frac{1}{4}\left[1 + O(\frac{1}{\sqrt{N}})\right] \max_{i=1,\cdots,4} \frac{C_i}{(i+1)}$$

traffic can be supported. Because of this decrease in the link capacities, in general, η^* may not be achievable but $\eta^* + O(1/N)$ can be achieved. The achievability part of the theorem follows.

As a note, it can be observed that τ-MPR is the schedule that maximizes the total number of successful transmissions in the network. This is true exactly if e_τ divides $\sqrt{N}$. Otherwise, τ-MPR provides a good approximation of the schedule maximizing the total number of successful transmissions in the network. To see this, let's look at the simpler case where the τ-MPR pattern can tile all the network. In τ-MPR, there are $N/(\tau+1)$ receivers in the network each receiving τ packets. Thus, the total number of successful transmissions in the network is $NC_\tau/(\tau+1)$. This quantity divided by $\bar{L}N$ is shown to upper bound the ξ in (54). Hence, τ-MPR is the schedule maximizing ξ and, equivalently, maximizing the total number of successful transmissions in the network. ■

In large networks topology discovery may not be feasible, and nodes may not be able use shortest routes. Similarly, during network initialization nodes spend some time discovering the network and may not be able to use the optimal routes. Gossiping [51], flooding, and random walking packets are routing alternatives in such networks that require nodes to know their neighbors, but not the whole network topology. In random walk, packets are relayed at each consecutive hop to a randomly chosen neighbor with uniform probabilities. If the network is connected, every packet eventually reaches its destination although the delivery may take a long time. In the proof showing the achievability of η we have used shortest path routing which gives average path length $\bar{L}$ proportional to $\sqrt{N}$. In [52], [53], it is shown that in Manhattan networks with random walking packets the average number of relays needed is of the order $N \log N$. Using an argument similar to the one in Theorem 4, it can be easily seen that the achievable rates with random walk routing is $O(\frac{1}{N \log N})$ whereas the capacity scales as $\frac{1}{\sqrt{N}}$. This result shows that the cost of lacking (or not using) topology information can be very high in large networks.

6.3 Capacity with Slotted ALOHA

In a distributed wireless network, topology specific scheduling may not be implementable in practice. On the other hand, it is important to quantify the performance loss due to using a sub-optimal, but easily implementable MAC protocols such as slotted ALOHA. In the next theorem we will give the highest rate achievable with the slotted ALOHA. We will consider the capacity problem where every node has infinitely many packets waiting in its queue to be delivered to the other nodes in the network. We will call rate $\lambda > 0$ *uniformly-achievable* if $(\lambda 1(i \neq j) : i, j \in \mathcal{N})$ is achievable with slotted ALOHA. The version of the slotted ALOHA we will consider is the one outlined in Section 3.1. In each slot every node randomly and independently makes a transmission decision. A node chooses to transmit a packet with *retransmission probability* q, the neighbor to be transmitted is chosen with uniform probabilities. If node i decides to transmit over link l, it chooses a route P with probability $Q(P,l)$. Then, if node i has a packet with route P, it is transmitted over link l. If node i doesn't have a packet with route P, a packet with source i and with destination $r(l)$ is transmitted. Symmetric, shortest path routing is used.

Theorem 5 (The capacity with slotted ALOHA) *The capacity of a Manhattan Network with slotted ALOHA is*

$$\eta_{ALOHA} = \frac{1}{4\bar{L}} \max_{0 \leq q \leq 1} \sum_{k=1}^{4} \binom{4}{k} q^k (1-q)^{5-k} C_k. \tag{56}$$

With the retransmission probability

$$q_{max} = \arg\max_{0 \leq q \leq 1} \sum_{k=1}^{4} \binom{4}{k} q^k (1-q)^{5-k} C_k,$$

every rate $\lambda < \eta_{ALOHA}/(N-1)$ is uniformly achievable with slotted ALOHA. There does not exist any q achieving rates $\lambda > \eta_{ALOHA}/(N-1)$ uniformly.

Proof In the description of ALOHA operation, we have assumed that if a node decides to transmit over link l but doesn't have a packet to transmit, then it transmits a packet with source $t(l)$ and destination $r(l)$. Because of this property the network becomes identical to the heavy loaded network described in Section 4.3. Due to Lemma 3, if rate λ is feasible, then $(1-\epsilon)\lambda$ is achievable for every ϵ.

Next, we will argue that rates below $\eta_{ALOHA}/(N-1)$ are uniformly achievable. We will consider an arbitrary link l in the network, and compute the expected number of successfully transmitted packets over that link. In order to receive a packet, node $r(l)$ must stay in the reception mode (this is with probability $1-q$), and the node $t(l)$ must transmit a packet to node $r(l)$ (this is with probability $q/4$). The probability that $0 \leq k' \leq 3$ other neighbors of node $r(l)$ transmit is $\binom{3}{k'} q^{k'} (1-q)^{3-k'}$. Given that node $r(l)$ does not transmit, $t(l)$ transmits a packet to node $r(l)$, and k' other neighbors transmit, the probability of success over link l is $\tilde{C}_{k'+1,1}$. Therefore, the expected number of successfully transmitted packets from node $t(l)$ to node $r(l)$ is

$$(1-q)\frac{q}{4} \sum_{k'=0}^{3} \binom{3}{k'} q^{k'} (1-q)^{3-k'} \tilde{C}_{k'+1,1}.$$

	$M=1$	$M=2$	$M=3$	$M=4$
$\eta \cdot \sqrt{N}$	1.00	1.33	1.50	1.60
$\eta_{ALOHA} \cdot \sqrt{N}$	0.16	0.34	0.46	0.50

Table 2: Capacity vs. Slotted ALOHA Capacity

Equation (39) gives that $\tilde{C}_{k'+1,1} = C_{k'+1}/(k'+1)$. By setting $k = k'+1$ in the above equation, we obtain the expected number of successful transmissions from $t(l)$ to $r(l)$ as

$$\Pr(l \in \mathcal{F}(t)) = \frac{1}{16}\sum_{k=1}^{4}\binom{4}{k} q^k (1-q)^{5-k} C_k. \tag{58}$$

Since a symmetric, shortest path routing is used, λ satisfying

$$\frac{(N-1)\lambda\bar{L}}{4} \leq \frac{1}{16}\sum_{k=1}^{4}\binom{4}{k} q^k (1-q)^{5-k} C_k. \tag{59}$$

is uniformly achievable. Maximizing (59) with respect to q, we see that all $\lambda < \eta_{ALOHA}/(N-1)$ are uniformly achievable.

For the converse see Appendix J. ■

Silvester and Kleinrock [19] obtained a special case of the above result under the collision channel assumption. Later, again using the collision channel, Tsybakov and Bakirov [13] showed that η_{ALOHA} is the maximum packet arrival rate at which the network can be stabilized with slotted ALOHA.

Omitting an additional $O(1/N)$ factor, rewrite η and η_{ALOHA} as

$$\begin{aligned} \eta &\simeq \frac{1}{\sqrt{N}} \max_{i=1,\cdots,4} \frac{2C_i}{i+1} \\ \eta_{ALOHA} &\simeq \frac{1}{\sqrt{N}} \max_{0\leq q\leq 1} \sum_{k=1}^{4}\binom{4}{k} q^k (1-q)^{5-k} \frac{C_k}{2}. \end{aligned}$$

The above expressions show that the scaling law is $O(1/\sqrt{N})$ and the per node throughput of the network goes to 0 both with optimal scheduling and slotted ALOHA. The main reason behind this fact is the uniform traffic pattern which gives average path length of $\bar{L} = O(\sqrt{N})$. This is similar to the capacity law observed in [1]. Another factor affecting the capacity is the performance of the MAC protocol which only affects the coefficient of the capacity but not the scaling law. As a numerical example, consider the MPR matrix for M-collision channel, $\mathbf{C}_M$. For $M \in \{1,2,3,4\}$, η and η_{ALOHA} are given in Table 2. It is seen that having the best MPR channel C_4 gives only 1.6 times improvement in η over the conventional collision channel C_1. On the other hand, in the collision channel (first column in Table 2), using optimal scheduling instead of slotted ALOHA provides about 6 times improvement.

6.4 Manhattan Networks with Fading Links

Suppose that each link of the Manhattan network is ON with probability p and OFF with probability $1-p$. (Here, we mean undirected links; the links (i,j) and (j,i) are always in the same state.) Assume that the network policy does not know which links are ON or OFF, and the nodes transmit their packets *without* knowing if their link is ON or OFF. This will be called a network without link state information (LSI).

Suppose that node i transmits to node j. If the link (i,j) is ON, and if j is the only transmitter in i's neighborhood whose link with i is ON, then the transmission is successful; it is unsuccessful otherwise. This channel can be expressed using an MPR matrix

$$\mathbf{C}_p = \begin{pmatrix} 1-p & p & & & \\ 1-2p(1-p) & 2p(1-p) & 0 & & \\ 1-3p(1-p)^2 & 3p(1-p)^2 & 0 & 0 & \\ 1-4p(1-p)^3 & 4p(1-p)^3 & 0 & 0 & 0 \end{pmatrix}.$$

In the MPR matrix, the entry $C_{k,1}$ is the probability that k neighbors transmit and one of them gets through, which is the case only when one link is ON and the rest $k-1$ are OFF; the probability of this event is $\binom{k}{1}p(1-p)^{k-1}$.

For this channel, Theorem 4 gives the network capacity as

$$\eta \simeq \frac{1}{\sqrt{N}} \max_{i=1,\cdots,4} \frac{i(1-p)^{i-1}}{(i+1)} 2p.$$

Theorem 4 also gives a way to schedule packets optimally. The value of

$$\begin{aligned} \tau &= \arg\max_{i=1,\cdots,4} \frac{i(1-p)^{i-1}}{(i+1)} \\ &= \begin{cases} 1 & 1 \geq p \geq \frac{1}{4} \\ 2 & \frac{1}{4} \geq p \geq \frac{1}{9} \\ 3 & \frac{1}{9} \geq p \geq \frac{1}{16} \\ 4 & \frac{1}{16} \geq p \geq 0 \end{cases} \end{aligned} \tag{60}$$

determines which τ-MPR pattern (Figure 5) to use as a function of severity of fading. From (60), it is apparent that one should use higher τ's when p is smaller. Using higher τ for small p can be considered as a special case of multiuser diversity. For instance when p is very small, in the neighborhood of receiver it is a very small probability that there is more than a single link ON. Therefore, 4-MPR scheduling (namely, "all neighbors transmit to the node in the center" strategy) does not lead to frequent collisions and increases the probability of successful transmission. The rates achievable with τ-MPR scheduling, $\tau \in \{1,2,3,4\}$, are shown in Figure 7.

It is an interesting question to ask what improvement one could obtain by having and exploiting the LSI. In case of LSI, the optimal policy again follows a similar idea: given the fading configuration, find and use the transmission schedule that maximizes the number of successful transmissions. However, in this case it is very hard to compute the achievable rates since there are numerous fading configurations. The following theorem gives some bounds on the capacity with LSI.

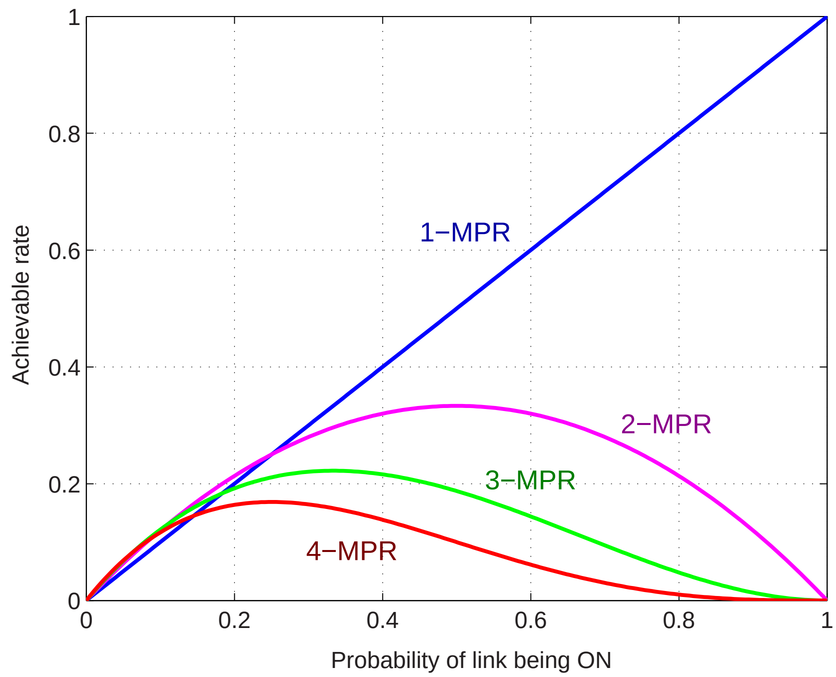

Figure 7: τ-MPR curve shows $\frac{\tau(1-p)^{\tau-1}}{\tau+1}2p$. This is $\sqrt{N}$ times the rate achievable with τ-MPR scheduling in a Manhattan network with fading links. Upper envelope of these curves is $\eta \cdot \sqrt{N}$. The performance of scheduling patterns is opposite for small p, that is, 4-MPR gives the highest throughput and 1-MPR gives the lowest.

Theorem 6 (The capacity of Manhattan Networks with LSI) *Let $\eta^{\#}$ be the capacity of the Manhattan network with LSI. Then,*

$$1 \leq \frac{\eta^{\#}}{\eta} \leq 2.86 + O(1/\sqrt{N}). \tag{61}$$

Moreover,

$$\lim_{\substack{N\to\infty \\ p\to 0}} \frac{\eta^{\#}}{\eta} = 2.5 \quad \textit{and} \quad \lim_{\substack{N\to\infty \\ p\to 1}} \frac{\eta^{\#}}{\eta} = 1. \tag{62}$$

Proof We will first discuss the extreme cases $p \simeq 0,\ p \simeq 1$. In these two regimes the results are easy to understand since there are simple strategies with performance close to the optimal.

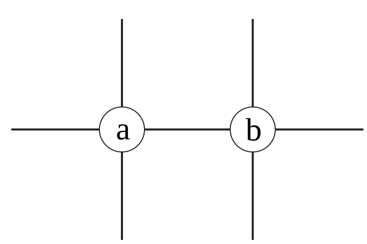

Figure 8: Scheduling example with LSI

If $p \simeq 0$, then very few links are ON, and the optimal strategy is transmitting over almost every ON. We describe a strategy that will be called *all ON's scheduled* next. Let a and b be two nodes in the network (Figure 8). In every slot, schedule a transmission over link (a,b) if and only if the link between a and b is ON, and all of the other six links connecting a and b to their respective neighbors are OFF. Choose the direction of transmission randomly; a to b with probability 1/2, and b to a with probability 1/2. With this scheduling the traffic which can be carried in each direction is $p(1-p)^6/2$ (This is the probability that the link is scheduled in a direction). Using a symmetric, shortest path routing we see that rates less than $2p(1-p)^6/(N-1)\bar{L}$ are uniformly achievable. Furthermore, rates above $2p/(N-1)\bar{L}$ are not uniformly achievable. This is true, since there are total $2N$ (undirected) links in the network and the expected number of ON links is $2pN$. Therefore, the transport capacity is upper bounded by $2pN$, and we have $\eta^{\#} \leq 2p/\bar{L}$ as a result of Theorem 3. Hence, we have just shown that

$$\frac{2p(1-p)^6}{\bar{L}} \leq \eta^{\#} \leq \frac{2p}{\bar{L}} \tag{63}$$

holds. When all sides are divided by

$$\eta = \frac{4p(1-p)^3}{5\bar{L}} \left[1 + O(\frac{1}{\sqrt{N}})\right],$$

we get

$$\frac{5(1-p)^3}{2} \left[1 + O(\frac{1}{\sqrt{N}})\right] \leq \frac{\eta^{\#}}{\eta} \leq \frac{5}{2(1-p)^3} \left[1 + O(\frac{1}{\sqrt{N}})\right].$$

The left hand side of (62) follows when we take the limit $p \to 0, N \to \infty$.

"All ON's scheduled" strategy almost achieves the capacity with LSI, which is $\eta^{\#} \simeq \frac{2p}{\bar{L}}$ for $p \simeq 0$. However, without LSI, the optimal strategy is 4-MPR scheduling which uses only 2/5 of the available links (This fact can be seen by counting the number of used links in Figure 5.d). This gives the "all ON's scheduled" strategy an advantage of 2.5 times over 4-MPR scheduling.

Next, we will look at the regime $p \simeq 1$. Note that the capacity with LSI is always less than the capacity without fading, that is, $\eta^{\#} \leq 1/2\bar{L}$. Moreover, the capacity with LSI is greater than capacity without LSI, that is, $\eta \leq \eta^{\#}$. Hence, the following holds

$$\frac{p}{2\bar{L}} \left[1 + O(\frac{1}{\sqrt{N}})\right] = \eta \leq \eta^{\#} \leq \frac{1}{2\bar{L}},$$

when $p > 1/4$. Divide all sides by η,

$$1 \leq \frac{\eta^{\#}}{\eta} \leq \frac{1}{p} \left[1 + O(\frac{1}{\sqrt{N}})\right].$$

Taking the limit $p \to 1, N \to \infty$ gives the right hand side of (62). One conclusion of this result is that if $p \simeq 1$ then almost all links are always ON, and with LSI using $1-MPR$ is almost optimal. The results for $p \simeq 0,\ p \simeq 1$ also suggest that the knowledge of LSI is more valuable when p is small. For high values of p, LSI is less important; one can simply use 1-MPR scheduling.

Next, we will upper bound the transport capacity for an arbitrary p. The considered network has multiple states and each successfully transmitted packet moves one distance unit. Without loss of generality assume that each link l in $\mathcal{E}_v$ (equation (15)) is successful. Hence, the transport capacity is equal to

$$\sum_{v\in\mathcal{V}} p(v)\sum_{l\in\mathcal{L}} d(l)\Pi(l;\mathcal{E}_v,v) \;=\; \sum_{v\in\mathcal{V}} p(v)\sum_{l\in\mathcal{L}} 1(l\in\mathcal{E}_v). \tag{65}$$

Observe the following equality

$$\sum_{l\in\mathcal{L}} 1(l\in\mathcal{E}_v) \;=\; \frac{1}{2}\sum_{i\in\mathcal{N}} 1((i,j)\in\mathcal{E}_v \text{ or } (j,i)\in\mathcal{E}_v \text{ for some } j). \tag{66}$$

The factor 1/2 comes due to the fact that each successfully transmitted packet is counted twice; once at the transmitter, and another time at the receiver. So, we can write the transport capacity as

$$\sum_{v\in\mathcal{V}} p(v)\frac{1}{2}\sum_{i\in\mathcal{N}} 1((i,j) \text{ or } (j,i)\in\mathcal{E}_v \text{ for some } j) \tag{67}$$

$$= \frac{1}{2}\sum_{i\in\mathcal{N}}\sum_{v\in\mathcal{V}} p(v)1((i,j) \text{ or } (j,i)\in\mathcal{E}_v \text{ for some } j). \tag{68}$$

The final sum $\sum_{v\in\mathcal{V}} p(v)1((i,j) \text{ or } (j,i)\in\mathcal{E}_v \text{ for some } j)$ is nothing but the expected number of successfully transmitted or received packets by node i. Since every node has four neighbors, this expectation is less than or equal to $1-(1-p)^4$ that is the probability that at least one out of four links is ON. Thus, we have proved the following upper bound on the transport capacity

$$\begin{aligned}\sum_{v\in\mathcal{V}} p(v)\sum_{l\in\mathcal{L}} 1(l\in\mathcal{E}_v) \;&\le\; \frac{1}{2}\sum_{i\in\mathcal{N}}(1-(1-p)^4)\\ &=\; \frac{N}{2}(1-(1-p)^4).\end{aligned} \tag{69}$$

As a result of Theorem 3, $\eta^{\#}\le(1-(1-p)^4)/2\bar{L}$. When we divide by η,

$$\begin{aligned}\frac{\eta^{\#}}{\eta} \;&=\; \frac{\eta^{\#}}{\eta^*(1+O(1/\sqrt{N}))}\\ &\le\; \left[\max_{i=1,\cdots,4}\frac{i(1-p)^{i-1}}{(i+1)}\right]^{-1}\frac{1-(1-p)^4}{2p}+O(\frac{1}{\sqrt{N}}).\end{aligned}$$

The last quantity is upper bounded by $2.853..+O(1/\sqrt{N})$, which is achieved at $p=0.1111...$ Therefore, (61) follows. ■

6.5 Other Regular Topologies and Optimal Connectivity

In this subsection we will look at the optimal connectivity problem in Manhattan and ring networks. We will use the intuition that the transport capacity provides tight upper bounds on network capacity in regular topologies. In particular, we will look for ways to increase the transport capacity by increasing connectivity.

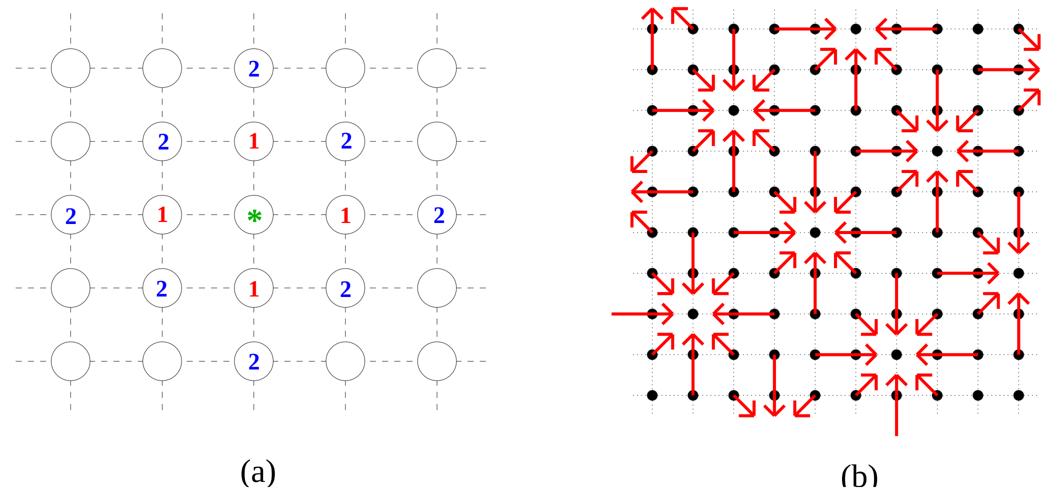

Figure 9: (a) This figure shows the neighbors of a node in 2-hop connected Manhattan network. Consider the node in the center (marked with $*$). It has a total of 12 neighbors, four of them are 1-hop neighbors (marked with 1), and eight are 2-hop neighbors (marked with 2). (b) 8-MPR scheduling in a Manhattan network with two hop connectivity. The network is divided into groups of 13 nodes. In each group the node in the center is the receiver, the receiver's two-hop neighbors are transmitters (there are eight such nodes in each group), and the 1-hop neighbors of each receiver stay idle (there are four such nodes in each group).

First, consider a Manhattan network with 2-hop connectivity, that is, every node is connected to neighbors two hops or one hop away (there are twelve such neighbors), see Figure 9.a. Consider the scheduling pattern in Figure 9.b. This pattern can be used under the assumption that the nodes can perfectly receive 8 packets simultaneously (*i.e.,* the MPR matrix is $\mathbf{C}_8$). When we tile the network with such a pattern, approximately 8/13 of the nodes are transmitters and the 1/13 nodes are receivers. Each transmitted packet moves 2 distance units, and the expected progress (rate-distance product) with this scheduling is

$$\frac{16}{13}N + O(\sqrt{N}). \tag{70}$$

This quantity is higher than the expected progress with the 4-MPR pattern in a 1-hop connected network (*i.e.,* the network considered in previous subsections). In 4-MPR scheduling, approximately 1/5 of the nodes are scheduled as receivers and each receiver receives 4 packets moving a single distance unit. Hence, the expected progress with 4-MPR scheduling is

$$\frac{4}{5}N + O(\sqrt{N}).$$

In the next theorem, we will show that (70) is the transport capacity of 2-hop connected Manhattan network with $\mathbf{C}_8$. We will also argue that the network capacity can be achieved using 8-MPR scheduling. These results show that if nodes can successfully receive 8 packets or more simultaneously, then 2-hop connected Manhattan network has higher capacity than the 1-hop connected Manhattan network.

Theorem 7 *Let η_{2-HOP} be the capacity of a 2-hop connected Manhattan network with MPR matrix $\mathbf{C}_8$. Then,*

$$\eta_{2-HOP} = \frac{16}{13}\frac{1}{\bar{L}} + O(\frac{1}{N}) \quad (71)$$

This quantity is about 54% higher than $\eta = \frac{4}{5}\frac{1}{\bar{L}} + O(\frac{1}{N})$, the capacity of a 1-hop connected Manhattan network with $\mathbf{C}_8$.

Proof First, we will show that (70) is the transport capacity, which proves $\eta_{2-HOP} \leq 16/13\bar{L}$ as a result of Theorem 3. In the considered network there exists only a single state. Consider a transmission schedule $\mathcal{E}$. In $\mathcal{E}$, let A_{i,j_1,j_2} be the number of nodes hearing i transmissions whose j_i are intended for them from their i-hop neighbors, $i = 1, 2$. Observe that i, j_1, j_2 must lie in

$$\mathcal{J} = \{(i, j_1, j_2) \in \mathcal{Z}_+^3 : i \leq 8, j_1 \leq 4, j_1 + j_2 \leq i\}.$$

A_{i,j_1,j_2} must satisfy

$$\sum_{(i,j_1,j_2)\in\mathcal{J}} A_{i,j_1,j_2}(13 - i + j_1 + j_2) \leq N, \quad (72)$$

since the total number of nodes is less than N. With these definitions, the transport capacity is equal to

$$\frac{1}{\bar{L}N}\sum_{l\in\mathcal{L}} d(l)\Pi(l;\mathcal{E}, v) = \frac{1}{\bar{L}N}\sum_{(i,j_1,j_2)\in\mathcal{J}} A_{i,j_1,j_2}(j_1 + 2j_2). \quad (73)$$

Maximizing (73) under the constraints (72), and A_{ij} being non-negative and real gives

$$\frac{1}{\bar{L}N}\sum_{l\in\mathcal{L}} d(l)\Pi(l;\mathcal{E}, v) \leq \frac{1}{\bar{L}N}\max_{(i,j_1,j_2)\in\mathcal{J}}\left\{N\frac{j_1 + 2j_2}{13 - i + j_1 + j_2}\right\} = \frac{16}{13}\frac{1}{\bar{L}}. \quad (74)$$

An inspection of the above inequality yields the fact that the transport capacity is less than or equal to $16N/13$. This means that (70) is the transport capacity. We also see that $\eta_{2-HOP} \leq 16/13\bar{L}$.

The achievability of η_{2-HOP} essentially uses the same idea we used in proving achievability of η in 1-hop Manhattan networks. We will briefly identify the differences and similarities. In a 2-hop connected network, it is advantageous for packets to move 2 hops per transmission rather than 1-hop. The routing vector achieving η_{2-HOP} uses this idea; packets keep jumping 2 hops until they reach to the destination or to a 1-hop neighbor of the destination after which they have to jump a single hop. We can balance the traffic load over network links in a way similar to Lemma 6. In a load balanced network 8-MPR pattern (and its shifted versions) can be used to achieve η_{2-HOP}. ■

Next, we will consider the ring networks. We assume that the nodes are placed on a ring with unit circumference with equal spacing (see Figure 10). The distance between node i and node j is defined as $d(i, j) = |i-j|/N$, where $|i-j| = i-j \mod N$. We will consider the following simple reception model which is an extension of the collision channel: If $\{(i_k, j_k) : k = 1, 2, \cdots, r\}$ is a set of transmitter-receiver pairs then the transmission from i_k to j_k is successful if

$$|i_l - j_k| > |i_l - j_l| \quad \text{for all } l \neq k.^9 \quad (75)$$

[9] The results in this section can be easily generalized to the case where $|i_l - j_k| > (1 + \Delta)|i_l - j_l|$, $\Delta > 0$, is used instead of the condition in (75). This is called the *protocol model* by Gupta and Kumar [1].

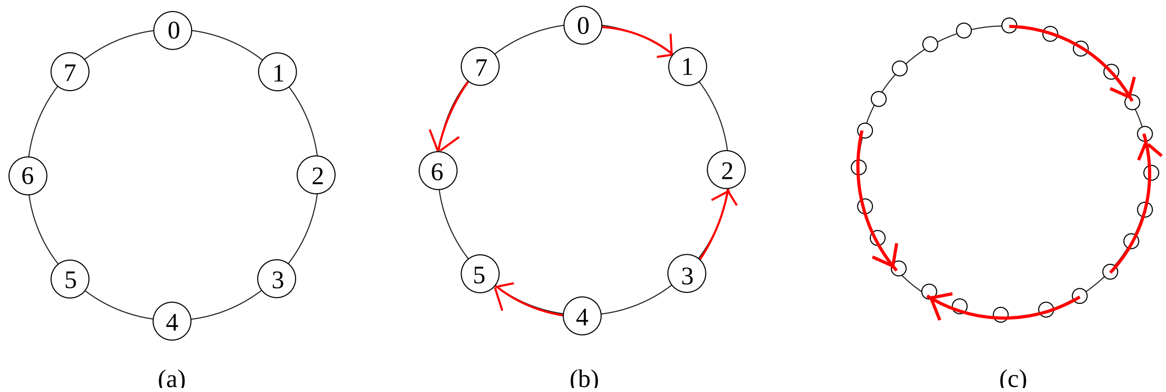

Figure 10: (a) A ring network with 8 nodes. The circumference is of unit length, and the distance between any two neighbors is 1/8 units. (b) 1-RING scheduling in a ring with 8 nodes. (c) 4-RING scheduling in a ring with 24 nodes.

If the nodes are restricted to communicate with two nearest neighbors, we call the network a 1-*hop connected ring*. In case the nodes are not restricted to 1-hop communication, we call the network a *free ring*. We have the following relation between capacities of 1-hop connected ring and the free ring.

Theorem 8 *The capacity of* 1*-hop connected ring is*

$$\eta_{1-RING} = \frac{2}{N} + O(\frac{1}{N^2}). \tag{76}$$

On the other hand, let τ_N be a sequence of integers satisfying

$$\lim_{N\to\infty} \tau_N = \infty, \qquad \lim_{N\to\infty} \frac{\tau_N}{N} = 0. \tag{77}$$

The capacity of free ring is

$$\eta_{F-RING} = \frac{4}{N} + O(\frac{1}{N\tau_N}). \tag{78}$$

Proof We'll start by computing the average path length $\bar{L}$ in a ring. For simplicity first assume that N is odd.

$$\begin{aligned}
\bar{L} &= \frac{1}{N-1}\sum_{i=1}^{N-1} d(0,i) \\
&= \frac{1}{(N-1)}\sum_{i=-(N-1)/2}^{(N-1)/2} \frac{|i|}{N} \\
&= \frac{2}{N(N-1)}\sum_{i=0}^{(N-1)/2} i \\
&= \frac{2}{N(N-1)}\frac{N^2-1}{8} \\
&= \frac{N+1}{4N} \\
&= \frac{1}{4} + O(\frac{1}{N})
\end{aligned}$$

A similar computation for N even yields the same formula $\bar{L} = 1/4 + O(1/N)$.

The proof of the theorem follows the methodology we used in Theorems 4 and 7. The transport capacities will be computed and used to upper bound the network capacities. To show the achievability of rates below the network capacity we will provide routing and medium access schemes.

Consider a family of scheduling patterns called τ-RING scheduling $\tau = 1, 2, \cdots, N$. In τ-RING scheduling the scheduled transmitter-receiver pairs are

$$\{(0, \tau), (2\tau + 1, \tau + 1), (2(\tau + 1), 2(\tau + 1) + \tau), 3(\tau + 1) + \tau, 3(\tau + 1)), \cdots\},$$

where this list is truncated at the point where either a transmitter or a receiver index goes above $N - 1$. Examples of the general τ-RING scheduling and 1-RING scheduling are shown in Figures 10.b and 10.c. Notice that 1-RING scheduling can be viewed as the one-dimensional analogue of 1-MPR scheduling. As we did in τ-MPR scheduling, we need to use the shifted versions of the τ-RING pattern so that each link can support equal rates.

The transport capacity of a 1-hop connected ring is

$$\frac{1}{2} + O(\frac{1}{N}). \tag{79}$$

This is achieved by 1-RING scheduling. To see the achievability observe that in 1-RING scheduling $N/2 + O(1)$ packets are transmitted, and each packet moves $1/N$ units; thus, the expected progress is

$$\frac{1}{N}\left(\frac{N}{2} + O(1)\right) = 1/2 + O(\frac{1}{N}).$$

To show equation (79) is an upper bound on the transport capacity, analogous arguments in Manhattan networks (in Theorem 4) can be used. In 1-hop connected ring since the transport capacity is (79) and $\bar{L} = 1/4 + O(1/N)$, it follows from Theorem 3 that

$$\eta_{1-RING} \leq \frac{2}{N} + O(\frac{1}{N^2}).$$

Since the circumference of the ring is unit length, the transport capacity of free ring is less than or equal to 1. Therefore,

$$\eta_{F-RING} \leq \frac{4}{N}. \tag{80}$$

Next, we will see that τ-RING scheduling arbitrarily closely approaches to the transport capacity. In τ_N-RING scheduling there are $N/(\tau_N + 1) + O(\tau_N/N)$ transmissions, and each packet moves τ_N/N unit. This means that the expected progress is

$$\frac{\tau_N}{\tau_N + 1} + O(\frac{\tau_N^2}{N^2}) = 1 + O(\frac{1}{\tau_N}), \tag{81}$$

where the equality follows from (77).

To achieve rates η_{1-RING} and η_{F-RING}, shortest path routing is used. Namely, in τ_N-RING scheduling, the packets move jumping τ_N-hops at each transmission, until they reach the destination or a neighbor of destination which is closer than τ_N hops. Balancing of the traffic is natural property of this routing such that almost all the traffic from a node i is directed to nodes $i + \tau_N$ and $i - \tau_N$. Using arguments similar to the achievability in Theorem 4, η_{F-RING} is achieved using shortest path routing and τ_N-RING scheduling. η_{1-RING} is achieved by putting $\tau_N = 1$ for all N. ∎

7 Conclusion

In this paper, we considered a general probabilistic model for wireless networks, and studied the network stability as well as the network capacity. We have characterized the stability and capacity regions using network flows. We have also introduced a class of policies sufficient to achieve stability and capacity. In the considered model the capacity and stability regions are not identical in general. However, we have given a mild condition under which the stability and the capacity regions are the same. We have also provided a simple necessary condition for achievability using the transport capacity.

In the second part of this paper, we applied the RTD policies and the flow analysis to the ring and Manhattan networks. We obtained a closed-form expression for the capacity of Manhattan networks and analyzed the impact of link fading, link state information and the topology information on achievable rates. We also compared a suboptimal scheme that uses ALOHA as its medium access to the optimal policy that jointly optimize medium access and routing. We finally examined the effect of variable connectivity radius on the capacity of Manhattan and ring networks.

The results for regular networks have ramifications for MAC in arbitrary networks. In Manhattan networks with multipacket receiving nodes the τ-MPR patterns (Figures 5 and 9.b), namely "neighboring nodes transmit into the center" strategy, was shown to be optimal for medium access. The τ-MPR scheduling locally resembles to an up-link especially for τ large. We expect this type of scheduling to be useful in arbitrary networks where multipacket reception is possible with multiple receive antennas or spread-spectrum. The 1-MPR scheduling and τ-RING scheduling (Figures 5 and 10), namely "transmitters turn each other their back and transmit" strategy, were shown to be optimal in networks without multipacket reception. We expect this idea to be useful in wireless networks with parts locally resembling to a one-dimensional topology. Examples include a wireless LAN in a corridor, or a group of nodes on a street or a highway.

There is an important open question related to the capacity of arbitrary networks: Does our analysis for regular networks carry over to arbitrary networks? In regular networks it is shown that the transport capacity provides tight upper bounds on the capacity. This upper bound is not always achievable in arbitrary networks. We however expect a general duality relation between the transport capacity and network capacity. To see what we mean by duality, recall that the transport capacity upper bound, inequality (13), is valid for all distance metrics satisfying the triangle inequality. Hence the upper bound minimized over all distance metrics is still an upper bound. We expect the minimized upper bound to be equal to capacity, at least under certain conditions such as the network to have a single state. This conjecture, if it is true, can be interpreted as duality, and may prove to be helpful in computing the capacity of arbitrary networks.

A Heavy Loaded Network - Proof of Proposition 2

Suppose that $\lambda_P > 0$ for some path $P = (s_0, s_1, \cdots, s_{m+1})$. Define $l_i = (s_i, s_{i+1})$ for $i = 0, 1, \cdots, m$. We will concentrate on P-packets in the heavy loaded network, look at the queue lengths of the nodes P-packets visit. To make the connections with Loynes's theory [44] clearer, we will use his notation. Define the following:

$$\begin{aligned} T_t^i &= F_{l_i,P}(t, D(t), v(t)) \\ S_{t+1}^0 &= A_{s_0,P}(t) \\ S_{t+1}^{i+1} &= F_{l_i,P}(t, D(t), v(t))1(n^*_{s_i,P}(t) > 0) \\ \omega_t^i &= n^*_{s_i,P}(t) - S_t^i. \end{aligned}$$

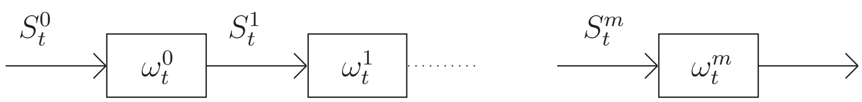

Figure 11: Series of queues

Using these definitions we obtain a new queue length processes (see Figure 11) where ω_t^i is the queue length of the new i'th queue, and the quantity S_{t+1}^{i+1} is the number of P-packets passing from the new queue s_i to the new queue s_{i+1} .

Claim The following relations follow from equation (18).

$$\begin{aligned} \omega_{t+1}^i &= [\omega_t^i + S_t^i - T_t]_+, && (82) \\ S_{t+1}^{i+1} &= \omega_t^i + S_t^i - \omega_{t+1}^i. && (83) \end{aligned}$$

Proof The P-packets leave the source node s_0 only through link l_0. Hence, the queue length process (18) at node s_0 follows

$$n^*_{s_0,P}(t+1) = n^*_{s_0,P}(t) + A_{s_0,P}(t) - F_{l_0,P}(t, D(t), v(t))1(n^*_{s_0,P}(t) > 0),$$

or equivalently,

$$n^*_{s_0,P}(t+1) = [n^*_{s_0,P}(t) - F_{l_0,P}(t, D(t), v(t))]_+ + A_{s_0,P}(t), \tag{84}$$

by rearranging,

$$F_{l_0,P}(t, D(t), v(t))1(n^*_{s_0,P}(t) > 0) = n^*_{s_0,P}(t) + A_{s_0,P}(t) - n^*_{s_0,P}(t+1). \tag{85}$$

The P-packets leave an intermediate node s_i, $i = 1, 2, \cdots, m$ only through link l_i, and they arrive at node s_i only from node s_{i-1}. Hence, the queue length process (18) for node s_i, $i = 1, 2, \cdots, m$ is

$$\begin{aligned} n^*_{s_i,P}(t+1) = {} & n^*_{s_i,P}(t) - F_{l_i,P}(t, D(t), v(t))1(n^*_{s_i,P}(t) > 0) \\ & + F_{l_{i-1},P}(t, D(t), v(t))1(n^*_{s_{i-1},P}(t) > 0), \end{aligned}$$

or equivalently,

$$\begin{aligned} n^*_{s_i,P}(t+1) = {} & [n^*_{s_i,P}(t) - F_{l_i,P}(t, D(t), v(t))]_+ \\ & + F_{l_{i-1},P}(t, D(t), v(t))1(n^*_{s_{i-1},P}(t) > 0), \end{aligned} \tag{86}$$

by rearranging,

$$F_{l_i,P}(t,D(t),v(t))1(n^*_{s_i,P}(t)>0)$$
$$= n^*_{s_i,P}(t)+F_{l_{i-1},P}(t,D(t),v(t))1(n^*_{s_{i-1},P}(t)>0)-n^*_{s_i,P}(t+1) \qquad (87)$$

The equation (82) follows from (84) for $i=0$, and from (86) for $i>0$. Similarly, the equation (83) follows from (85) for $i=0$, and from (87) for $i>0$. ■

The equations in [44] analogous to (82) and (83) form the basis of Loynes's stability theory. In our problem, equation (82) is the same as its analogue in Loynes [44], but equation (83) is different from its analogue [44, eqn. (13)]. Nonetheless, the difference is a minor one, and his analysis carries over without any difficulty. For stability, Loynes requires $\mathbb{E}\{S^0_t - T^i_t\} < 0$ for all i, which holds due to equation (22), and stationarity and ergodicity of the process $((S^0_t, T^0_t, T^1_t, \cdots, T^m_t) : t \in \mathcal{Z}_+)$, which is the case in our problem[10].

B Stochastic Ordering - Proof of Lemma 3

When two random vectors X and Y have the same distribution, we write $X \overset{d}{=} Y$. Let $\{0,1\}^{\mathcal{LC}}$ be the set of all vectors of the form $X = (X_{l,P} \in \{0,1\} : l \in \mathcal{L}, P \in \mathcal{C})$. For X, Y in $\{0,1\}^{\mathcal{LC}}$ define the product vector as

$$XY = (X_{l,P}Y_{l,P} : l \in \mathcal{L}, P \in \mathcal{C}).$$

Let X, Y be two $\{0,1\}^{\mathcal{LC}}$ valued random vectors such that

$$\Pr\{X \ge Z\} \le \Pr\{Y \ge Z\} \text{ for all } Z \in \{0,1\}^{\mathcal{LC}}.$$

X is said to be *smaller than Y in the usual stochastic order* (denoted by $X \le_{\text{st}} Y$).

There are several ways to look at stochastic ordering relations. One approach is provided by the definition above which does not restrict X and Y to be defined in the same probability space. However, there is another, sometimes more convenient, way of looking at stochastic order. If $X \le_{\text{st}} Y$, then this equivalent approach (given as Theorem 4.B.1 in [49]) constructs new random vectors $\hat{X}$ and $\hat{Y}$ in some probability space such that $\hat{X} \overset{d}{=} X$, $\hat{Y} \overset{d}{=} Y$ and $\hat{X} \le \hat{Y}$ with probability 1. In other words, we can view the stochastic order as the usual order in an appropriate probability space.

In the proof of Lemma 3, it is the second approach we will be using. We will construct two new stochastic processes (one mimicking the normal network, the other mimicking the heavy loaded one) such that the stochastic order relation given in Lemma 3 can be viewed as the usual order in some appropriate space. Our plan is as follows. We will first show the existence of this new space. Then we will observe the results in Lemma 3 in the new space, and discuss the equivalence of the newly constructed network and the real network.

One useful property of the stochastic order is provided below.

[10]This property is because of the fact that the processes describing the network are jointly stationary and ergodic (or in other words, metrically transitive [54]). The process $((S^0_t, T^0_t, T^1_t, \cdots, T^m_t) : t \in \mathcal{Z}_+)$ can be seen to be a measurable function of the underlying network processes which are stationary and ergodic by our assumption. As discussed in Doob [54], measurable functions of stationary and ergodic sequences are also stationary and ergodic.

Theorem 9 *Let X, Y_i be $\{0,1\}^{\mathcal{LC}}$ valued random vectors, and Z_i be a (deterministic) vector in $\{0,1\}^{\mathcal{LC}}$, $i = 1, 2, \cdots, r$. If*

$$XZ_i \leq_{st} Y_i Z_i \qquad i = 1, 2, \cdots, r$$

then there exists random vectors $\hat{X}, \hat{Y}_i$ defined on the same probability space such that $\hat{X} \overset{d}{=} X$, $\hat{Y}_i \overset{d}{=} Y_i$ and $\hat{X} Z_i \leq \hat{Y}_i Z_i$ with probability 1, $i = 1, 2, \cdots, r$.

Proof This theorem is a straightforward extension of Theorem 4.B.1 in [49], and its proof is omitted. ■

The next lemma is the major step in constructing the new network processes. Recall that for vectors E, F, D in $\{0,1\}^{\mathcal{LC}}$, the sets $\mathcal{E}, \mathcal{F}$ and $\mathcal{D}$ are defined as $\mathcal{E} = \{l \in \mathcal{L} : E_{l,P} = 1 \text{ for some } P \in \mathcal{C}\}$, $\mathcal{F} = \{l \in \mathcal{L} : F_{l,P} = 1 \text{ for some } P \in \mathcal{C}\}$, $\mathcal{D} = \{l \in \mathcal{L} : D_{l,P} = 1 \text{ for some } P \in \mathcal{C}\}$.

Lemma 7 *There exists a set of $\{0,1\}^{\mathcal{LC}}$ valued random vectors*

$$\mathcal{I} = \{I(E, D, v) : E, D \in \{0,1\}^{\mathcal{LC}}, E \leq D, v \in \mathcal{V}\},$$

defined on the same probability space, such that for every random vector $I(E, D, v) \in \mathcal{I}$,

i) $I(E, D, v)$ is distributed according to $\pi(\cdot\,; \mathcal{E}, v)$. That is, $\Pr\{I(E, D, v) = F\} = \pi(\mathcal{F}; \mathcal{E}, v)$ *for every vector F in $\{0,1\}^{\mathcal{LC}}$.*

ii) $I(D, D, v)E \leq I(E, D, v)$ with probability 1.

Proof Let X, Y, Z be vectors in $\{0,1\}^{\mathcal{LC}}$ such that $X \leq Y$. Observe that

$$ZY \geq X \text{ if and only if } Z \geq X. \tag{88}$$

Let $I(D, D, v), I(E, D, v)$ be random vectors satisfying (i). Observe that

$$I(E, D, v)E = I(E, D, v). \tag{89}$$

This is because $\Pr\{I(E, D, v) = F\} > 0$ only if $F \leq E$ which implies that $FE = F$.

We claim that

$$I(D, D, v)E \leq_{\text{st}} I(E, D, v)E. \tag{90}$$

To see this, consider a vector F in $\{0,1\}^{\mathcal{LC}}$ such that $F \leq E$.

$$\begin{aligned}
\Pr\{I(D, D, v)E \geq F\} &= \Pr\{I(D, D, v) \geq F\} \\
&= \Pi(\mathcal{F}; \mathcal{D}, v) \\
&\leq \Pi(\mathcal{F}; \mathcal{E}, v) \\
&= \Pr\{I(E, D, v) \geq F\} \\
&= \Pr\{I(E, D, v)E \geq F\}.
\end{aligned}$$

The first equality is due to (88). The second and third equalities are because $I(D,D,v), I(E,D,v)$ are random vectors satisfying (i). The last equality is due to (89). The inequality is because of $E \leq D$ and assumption (A1).

As a result of ordering in (90), we can apply Theorem 9. For this, fix some D and v. Set $X = I(D,D,v)$, $Y_i = I(E,D,v)$, $Z_i = E$ such that each $E \leq D$ corresponds to a different Z_i. The result of Theorem 9 says that there exists a probability space such that we can define random vectors $\{I(E,D,v) : E \leq D\}$ satisfying (i) and $I(D,D,v)E \leq I(E,D,v)E$. Also notice that $I(E,D,v)E = I(E,D,v)$ holds due to (89).

We have shown the existence of the space of random vectors $\{I(E,D,v) : E \leq D\}$ satisfying (i) and (ii) for fixed D and v. These sets of random vectors for different D and v can be put into the same probability space since the involved random vectors are discrete. Therefore, a set of random vectors $\mathcal{I}$ satisfying (i) and (ii) exists in some probability space and the lemma follows. ■

Next, we will define new stochastic processes. Suppose that we have a vector valued stochastic process $\{\mathcal{I}(t) : t \in \mathcal{Z}_+\}$ such that each entry

$$\mathcal{I}(t) = \{I(t,E,D,v) : E,D \in \{0,1\}^{\mathcal{LC}}, E \leq D, v \in \mathcal{V}\},$$

is i.i.d. distributed same as $\mathcal{I}$ in the previous lemma. We define the *normal m-network* with the following equation

$$\begin{aligned} m_{i,P}(t+1) = m_{i,P}(t) - &\sum_{l\in\mathcal{L}:\ t(l)=i} I_{l,P}(t,E(t),D(t),v(t)) \\ + \sum_{l\in\mathcal{L}:\ r(l)=i} &I_{l,P}(t,E(t),D(t),v(t)) \ + A_{i,P}(t). \end{aligned} \tag{91}$$

In the above equation the notation $m_{i,P}(t)$ refers to the queue length, and the rest are as before. The queue length processes in the normal network (16) and the normal m-network are indistinguishable. That is, it can be easily checked by induction over r that the joint distribution of queue lengths $\{(n_{i,P}(t) : i \in \mathcal{N}, P \in \mathcal{C}), t = 1,2,\cdots,r\}$ and $\{(m_{i,P}(t) : i \in \mathcal{N}, P \in \mathcal{C}), t = 1,2,\cdots,r\}$ are the same for all r. This result basically follows from property (i) of the previous lemma *i.e.,* each $I(t,E,D,v)$ is distributed according to $\pi(\ \ ;\mathcal{E},v)$. Since all joint queue length distributions are the same it follows that

$$\Pr\Big(\sum_{i\in\mathcal{N}} n_{i,P}(t) > \theta\Big) = \Pr\Big(\sum_{i\in\mathcal{N}} m_{i,P}(t) > \theta\Big) \tag{92}$$

for all θ. Moreover,

$$\frac{1}{t} n_{i,P}(t) \rightarrow 0 \text{ w.p.1 \ if and only if } \ \frac{1}{t} m_{i,P}(t) \rightarrow 0 \text{ w.p.1}, \tag{93}$$

as $t\rightarrow\infty$, where w.p.1 stands for with probability 1.

Similarly, we define the heavy loaded m-network as:

$$\begin{aligned} &m^*_{i,P}(t+1) = m^*_{i,P}(t) + A_{i,P}(t) \\ &- \sum_{l\in\mathcal{L}:\ t(l)=i} I_{l,P}(t,D(t),D(t),v(t))\ 1(m^*_{t(l),P}(t) > 0) \\ &+ \sum_{l\in\mathcal{L}:\ r(l)=i} I_{l,P}(t,D(t),D(t),v(t))\ 1(m^*_{t(l),P}(t) > 0), \end{aligned} \tag{94}$$

if $i \neq q(P)$. Just as the normal m-network mimics the behavior of the normal network, the heavy loaded m-network mimics the heavy loaded network (18). That is, the joint distribution of queue lengths $\{(n^*_{i,P}(t) : i \in \mathcal{N}, P \in \mathcal{C}), t = 1, 2, \cdots, r\}$ and $\{(m^*_{i,P}(t) : i \in \mathcal{N}, P \in \mathcal{C}), t = 1, 2, \cdots, r\}$ are the same for all r. Therefore, it is true that

$$\Pr(\sum_{i\in\mathcal{N}} n^*_{i,P}(t) > \theta) = \Pr(\sum_{i\in\mathcal{N}} m^*_{i,P}(t) > \theta) \tag{95}$$

for all θ. Moreover,

$$\frac{1}{t} n^*_{i,P}(t) \to 0 \text{ w.p.1 } \text{ if and only if } \frac{1}{t} m^*_{i,P}(t) \to 0 \text{ w.p.1}, \tag{96}$$

as $t\to\infty$.

Next, we will see that

$$\sum_{i\in\mathcal{N}} m^*_{i,P}(t) \ \geq \ \sum_{i\in\mathcal{N}} m_{i,P}(t), \tag{97}$$

holds with probability 1, for all P, t. This inequality is due to the property (ii) of the last lemma: if $E_{l,P}(t) \cdot I_{l,P}(t, D(t), D(t), v(t)) = 1$ for some i, P then $I_{l,P}(t, E(t), D(t), v(t)) = 1$. Observe that $E_{l,P}(t) \cdot I_{l,P}(t, D(t), D(t), v(t)) = 1$ if and only if $E_{l,P}(t) = 1$ (a P-packet is transmitted over link l in the normal m-network) and $I_{l,P}(t, D(t), D(t), v(t)) = 1$ (a P-packet is successfully transmitted over link l in the heavy loaded m-network if $m^*_{t(l),P}(t) > 0$). According to Property (ii) these two events imply that $I_{l,P}(t, E(t), D(t), v(t)) = 1$ (P-packet is successfully transmitted over link l in the normal m-network). That is, we can simply say that if a packet is successfully transmitted in the heavy loaded m-network in slot t, then in the normal m-network either a packet is successfully transmitted in slot t, or the transmitter queue is empty which means all packets have already been transmitted. This reasoning, by using induction over t, leads to (97).

Because of (97),

$$\Pr(\sum_{i\in\mathcal{N}} m^*_{i,P}(t) > \theta) \ \geq \ \Pr(\sum_{i\in\mathcal{N}} m_{i,P}(t) > \theta) \tag{98}$$

holds for all θ. Equations (98), (92) and (95) give (23).

In Lemma 2 we have shown that $\frac{1}{t} n^*_{i,P}(t) \to 0$ with probability 1 as $t\to 0$. Due to (96), we have $\frac{1}{t} m^*_{i,P}(t) \to 0$ with probability 1. Since the queue lengths are nonnegative processes, inequality (97) gives $\frac{1}{t} m_{i,P}(t) \to 0$ with probability 1. And, as a result of (93), it is true that $\frac{1}{t} n_{i,P}(t) \to 0$ with probability 1, as required.

C Proof of Lemma 4

Define $G_{ij}(t) = \sum_{l\in\mathcal{L}:\ r(l)=i} F_{lj}(t)$ and $H_{ij}(t) = \sum_{l\in\mathcal{L}:\ t(l)=i} F_{lj}(t)$. With these definitions, we can write equation (24) as

$$n_{ij}(T) = \sum_{t=0}^{T-1} \left[A_{ij}(t) + G_{ij}(t) - H_{ij}(t) \right]. \tag{99}$$

If the network is stable, the following is satisfied.

$$\lim_{\theta\to\infty} \limsup_{t\to\infty} \Pr\{n_i(t) > \theta\} = 0, \quad \text{for all } i \in \mathcal{N}.$$

Now, pick an $i \in \mathcal{N}$. For all $\theta, \epsilon > 0$, there exists t_0 such that for all $t > t_0$, $\Pr\{n_i(t) > t\epsilon\} \leq \Pr\{n_i(t) > \theta\}$. Consider the limit as $t \to \infty$,

$$\begin{aligned}
\limsup_{t\to\infty} \Pr\{n_i(t) > t\epsilon\} &\leq \limsup_{t\to\infty} \Pr\{n_i(t) > \theta\} \\
&\leq \lim_{\theta\to\infty} \limsup_{t\to\infty} \Pr\{n_i(t) > \theta\} \\
&= 0,
\end{aligned}$$

which means that $\frac{1}{t} n_i(t) \xrightarrow{\mathcal{P}} 0$, where $\xrightarrow{\mathcal{P}}$ denotes convergence in probability. Pick an arbitrary $j \in \mathcal{N}$. Since $n_{ij}(t)$ is nonnegative and less than $n_i(t)$, $\frac{1}{t} n_{ij}(t) \xrightarrow{\mathcal{P}} 0$. Write (99) as

$$\frac{1}{t}\sum_{r=0}^{t-1} A_{ij}(r) - \frac{1}{t}\sum_{r=0}^{t-1}[H_{ij}(r) - G_{ij}(r)] = \frac{1}{t} n_{ij}(t) \xrightarrow{\mathcal{P}} 0. \tag{101}$$

Since $(A_{ij}(t) : t \in \mathcal{Z}_+)$ is ergodic,

$$\lim_{t\to\infty} \frac{1}{t}\sum_{r=0}^{t-1} A_{ij}(r) = \lambda_{ij}$$

almost surely. The previous equation implies that

$$J_{ij}(t) \triangleq \frac{1}{t}\sum_{r=0}^{t-1}[H_{ij}(r) - G_{ij}(r)] \xrightarrow{\mathcal{P}} \lambda_{ij}. \tag{102}$$

Observe that $|H_{ij}(t)| \leq |\mathcal{L}|$ and $|G_{ij}(t)| \leq |\mathcal{L}|$ for all t. Therefore, $|J_{ij}(t)| \leq 2|\mathcal{L}|$. This, together with (102), imply that $\mathbb{E}J_{ij}(t) \to \lambda_{ij}$. When we take expectation of both sides in (101),

$$\frac{1}{t}\mathbb{E}n_{ij}(t) = \lambda_{ij} - \mathbb{E}J_{ij}(t) \to 0.$$

The lemma follows.

D Proof of Lemma 5

The key step in the proof is observing that

$$\mathbb{E}\{\sum_{j\in\mathcal{N}} F_{lj}(t) \mid \mathcal{E}(t) = \mathcal{E}, v(t) = v\} = \Pi(l; \mathcal{E}, v)$$

holds for all $\mathcal{E}, v, l, t$. This follows from the definition of $\Pi(\cdot)$. The following sequence of equalities lead to the required lemma. For every $l \in \mathcal{L}$,

$$\sum_{j\in\mathcal{N}} \mathbb{E}\{\frac{1}{T}\sum_{t=0}^{T-1} F_{lj}(t)\} = \frac{1}{T}\sum_{t=0}^{T-1} \mathbb{E}\{\sum_{j\in\mathcal{N}} F_{lj}(t)\}$$

$$
\begin{aligned}
&= \frac{1}{T}\sum_{t=0}^{T-1}\sum_{v\in\mathcal{V},\mathcal{E}\in\mathscr{E}} \mathbb{E}\{\sum_{j\in\mathcal{N}} F_{lj}(t) \mid \mathcal{E}(t)=\mathcal{E}, v(t)=v\} \\
&\qquad\qquad\qquad\qquad\qquad \Pr\{\mathcal{E}(t)=\mathcal{E}, v(t)=v\} \\
&= \frac{1}{T}\sum_{t=0}^{T-1}\sum_{v\in\mathcal{V},\mathcal{E}\in\mathscr{E}} \Pi(l;\mathcal{E},v)\Pr\{\mathcal{E}(t)=\mathcal{E}|v(t)=v\}p(v) \\
&= \sum_{v\in\mathcal{V},\mathcal{E}\in\mathscr{E}} \Pi(l;\mathcal{E},v)\frac{1}{T}\sum_{t=0}^{T-1}\Pr\{\mathcal{E}(t)=\mathcal{E}|v(t)=v\}p(v) \\
&= \sum_{v\in\mathcal{V},\mathcal{E}\in\mathscr{E}} \Pi(l;\mathcal{E},v)p(\mathcal{E};v)p(v).
\end{aligned}
$$

E Flow Vectors - Proof of Proposition 3

First, we will introduce some definitions and lemmas. The proof of Proposition 3 will be given at the end of this section. For a given network with nodes $\mathcal{N}$ and links $\mathcal{L}$, a vector $E = (e_l \geq 0 : l \in \mathcal{L})$ will be called a *flow vector*. For a given flow vector E, for every $i \in \mathcal{N}$ define

$$f_i(E) \triangleq \sum_{l\in\mathcal{L}:\ t(l)=i} e_l - \sum_{l\in\mathcal{L}:\ r(l)=i} e_l$$

as the *flow* from node i into the network.

A *loop* in a network is defined as an ordered m-tuple of links $(l_1, l_2, \cdots, l_m) \in \mathcal{L}^m$ such that the following is satisfied.

(i) $r(l_m) = t(l_1)$

(ii) $r(l_i) = t(l_{i+1}),\ i = 1, 2, \cdots, m-1$

(iii) $(r(l_1), r(l_2), \cdots, r(l_m))$ is a path.

Denote the set of all loops with $\mathscr{L}$. In a flow vector E, the flow across a loop $L \in \mathscr{L}$ is defined as

$$flow(L, E) \triangleq \min\{e_{l_1}, e_{l_2}, \cdots, e_{l_m}\}.$$

A flow vector *without loops* is a flow vector E such that for any loop L, $flow(L, E) = 0$.

Lemma 8 *Let $E = (e_l : l \in \mathcal{L})$ be a flow vector. There exists a flow vector without loops $\hat{E} = (\hat{e}_l : l \in \mathcal{L})$ satisfying the following*

(i) $0 \leq \hat{e}_l \leq e_l$, *for all* $l \in \mathcal{L}$.

(ii) The flow of each node in E and $\hat{E}$ are the same, i.e., $f_i(E) = f_i(\hat{E}),\ \forall i \in \mathcal{N}$.

Proof For a loop $L = (l_1, l_2, \cdots, l_m)$ and a link l, we say that $l \in L$ if $l = l_k$ for some k between 1 and m. In order to prove the lemma, we will give an algorithm which eliminates all loops in E step by step.

Suppose that $\mathscr{L} = \{L_1, L_2, \cdots, L_{|\mathscr{L}|}\}$. We will use k as an index variable. Initialize $k = 1$, and set $\hat{E} = E$. Then apply the following operation on the entries of $\hat{E}$,

$$\text{For all } l \in L_k, \text{ change } \hat{e}_l \text{ to } \hat{e}_l - flow(L_k, \hat{E}).$$

We can see that after this operation (i) and (ii) are satisfied. Next, increment k and continue this procedure for $k = 1, 2, \cdots, |\mathscr{L}|$ one by one. In the end, we not only end up with a vector $\hat{E}$ without loops, but also the final $\hat{E}$ satisfies (i) and (ii) since after each step they are satisfied. ■

Fix a $j \in \mathcal{N}$. Define

$$E_j = (e_{lj} : l \in \mathcal{L}). \tag{103}$$

In the flow vector E_j, if $\lambda_{ij} > 0$ holds for some node i, then $f_i(E_j) = \lambda_{ij} - \gamma_{ij}$ as given in (29). Furthermore, due to (31) and (32) if $\lambda_{ij} > 0$ then $f_i(E_j) > 0$. On the other hand, if $\lambda_{ij} = 0$ then $f_i(E_j) = -\gamma_{ij} \leq 0$. Hence, $\lambda_{ij} > 0$ if and only if $f_i(E_j) > 0$.

If $f_i(E_j) > 0$, node i will be called a *source node.* If $f_i(E_j) < 0$, node i will be called an *accumulation node.* If $V = (l_1, l_2, \cdots, l_n) \in \mathcal{L}^n$ and $l_{n+1} \in \mathcal{L}$ then define $V \oplus (l_{n+1}) = (l_1, l_2, \cdots, l_n, l_{n+1})$, $(l_{n+1}) \oplus V = (l_{n+1}, l_1, l_2, \cdots, l_n\}$. If $V = \phi$, then $V \oplus (l_{n+1}) = (l_{n+1}) \oplus V = (l_{n+1})$.

Lemma 9 *Let E_j be the flow vector defined in (103). There exists a flow vector without loops $\hat{E}_j = (\hat{e}_{lj} : l \in \mathcal{L})$ satisfying*

(i) $0 \leq \hat{e}_{lj} \leq e_{lj}$, *for all* $l \in \mathcal{L}$.

(ii) Except j, there does not exist any accumulation node in $\hat{E}_j$, i.e., for all $i \neq j$, $f_i(\hat{E}_j) \geq 0$.

(iii) If $i \in \mathcal{N}$ is a source node, then $f_i(\hat{E}_j) > \lambda_{ij} - \epsilon$.

Proof We will give an algorithm for obtaining $\hat{E}_j$ from E_j. First apply the algorithm in Lemma 8 and obtain a flow vector without loops $\hat{E}_j$ from E_j.

(A) Check if there exist an accumulation node $i \neq j$ in $\hat{E}_j$. If there is no other accumulation node, terminate.

(B) Set $V = \phi$.

(C) For node i, $f_i(\hat{E}_j) \leq 0$ is satisfied, and there exists a link l such that $r(l) = i$ and $e_l \neq 0$. Set V to $(l) \oplus V$.

  (a) If $t(l)$ is a source node then for all $l \in V$ change $\hat{e}_{lj}$ to $\hat{e}_{lj} - \min\{\hat{e}_{lj} : l \in V\}$. Go to step (A).

  (b) If $t(l)$ is not a source node then $f_{t(l)}(\hat{E}_j) \leq 0$. Set $i = t(l)$ and go to step (C).

In part (C), V does not form a loop at any time (*i.e.,* $V \notin \mathscr{L}$) since the flow vector does not contain any loops. Part (C) terminates in finite number of steps since there exists finitely many nodes that can be visited, and a node can not be visited more than once. The algorithm terminates in finite number of steps since there are finitely many nodes and paths in the network, and due to decrease of $\hat{e}_{lj}$'s in part (C)-(a), if a path is followed once, it can not be followed once more. We can check the properties (i) to (iii):

(i) Holds because at each step $\hat{e}_{lj}$ non-increases.

(ii) Holds since the algorithm eliminates all accumulation nodes except j.

(iii) Holds because $\sum_i \gamma_{ij} < \epsilon$, and the removal of γ_{ij} from the accumulation nodes decreases the flow from each source node at most $\sum_i \gamma_{ij}$.

■

Proof *(Proposition 3)* For each $j \in \mathcal{N}$ define E_j as in (103), and apply the algorithm in Lemma 9 to obtain $\hat{E}_j$. Initialize $(y_P = 0 : P \in \mathscr{P})$. For each $i, j \in \mathcal{N}, \lambda_{ij} - \epsilon 1(\lambda_{ij} > 0) > 0$, apply the following algorithm:

(A) If $f_i(\hat{E}_j) > 0$ continue, otherwise terminate.

(B) Set $V = \phi$, and $k = i$.

(C) $f_k(\hat{E}_j) \geq 0$ and there exists a link $l \in \mathcal{L}$ such that $t(l) = k$ and $\hat{e}_{lj} > 0$. Set V to $V \oplus (l)$.

  (a) If $r(l) = j$, then set $y_V = \min\{\hat{e}_{lj} : l \in V\}$. For all $l \in V$ change $\hat{e}_{lj}$ to $\hat{e}_{lj} - \min\{\hat{e}_{lj} : l \in V\}$. Go to step (A).

  (b) If $r(l) \neq j$, then set $k = r(l)$ and go to step (C)

Due to Lemma 9(ii), we can make sure that in part (C) there exists a link $l \in \mathcal{L}$ such that $t(l) = k$ and $\hat{e}_{lj} > 0$.

In the end, $(y_P : P \in \mathscr{P})$ generated by the algorithm satisfies the following:

(i) Due to Lemma 9(i), $\forall j \in \mathcal{N}$,

$$\sum_{i \in \mathcal{N}} \sum_{P \in \mathscr{P}_{ij}:\ l \in P} y_P \leq e_{lj}. \tag{104}$$

(ii) Due to Lemma 9(iii), $\forall i, j \in \mathcal{N}$,

$$\lambda_{ij} - \epsilon 1(\lambda_{ij} > 0) \leq \sum_{P \in \mathscr{P}_{ij}} y_P. \tag{105}$$

For each $i, j \in \mathcal{N}$, $P \in \mathscr{P}_{ij}$, define

$$x_P \triangleq \begin{cases} \frac{y_P}{\sum_{P \in \mathscr{P}_{ij}} y_P}, & \text{if } \lambda_{ij} - \epsilon 1(\lambda_{ij} > 0) > 0 \\ 0, & \text{otherwise.} \end{cases} \tag{106}$$

We can check that for each $j \in \mathcal{N}, l \in \mathcal{L}$,

$$\begin{aligned} \sum_{i \in \mathcal{N}} \sum_{P \in \mathscr{P}_{ij}:\ l \in P} x_P(\lambda_{ij} - \epsilon 1(\lambda_{ij} > 0)) &\leq \sum_{i \in \mathcal{N}} \sum_{P \in \mathscr{P}_{ij}:\ l \in P} y_P \\ &\leq e_{lj}. \end{aligned} \tag{107}$$

The first inequality follows from (105) and (106). The second one is due to (104). As a result of (107), $(x_P : P \in \mathscr{P})$ satisfies (33), and the proposition holds. ■

F Theorem 2: Achievability Implies Feasibility

With minor modifications, we can use the same techniques employed in proving the stability implies feasibility (Section 5).

If $\lambda = (\lambda_{ij} : i, j \in \mathcal{N})$ is achievable, then by Fatou's lemma, for all $i, j \in \mathcal{N}$,

$$\begin{aligned} \lambda_{ij} &\leq \mathbb{E}\left\{ \liminf_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} W_{ij}(t) \right\} \\ &\leq \liminf_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}\{W_{ij}(t)\}. \end{aligned}$$

Suppose that at time $T \in \mathcal{Z}_+$, for all $i, j \in \mathcal{N}$,

$$\lambda_{ij} - \epsilon 1(\lambda_{ij} > 0) \leq \frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}\{W_{ij}(t)\}.$$

To show that $\lambda - \epsilon 1_\lambda \geq 0$ is feasible for all $\epsilon > 0$, we can define the scheduling vector $G = (p(\mathcal{E}; v) : v \in \mathcal{V}, \mathcal{E} \in \mathscr{E})$ as in (27), and the other quantities e_{lk}, γ_{ik} similar to the ones defined in Section 5. Then, an algorithm almost identical to the one used in Appendix E can be used to construct H showing that $\lambda - \epsilon 1_\lambda$ is feasible.

G Proof of Proposition 4

The average distance a packet originating from node i travels is same for all i. As a result of this symmetry, we can compute $\bar{L}$ by averaging the distances between the node (0,0) and the other nodes in the network,

$$\begin{aligned} \bar{L} &= \frac{1}{N-1} \sum_{x=0}^{\sqrt{N}-1} \sum_{y=0}^{\sqrt{N}-1} d\{(0,0),(x,y)\} \\ &= \frac{1}{N-1} \sum_{x=0}^{\sqrt{N}-1} \sum_{y=0}^{\sqrt{N}-1} \min\{x, \sqrt{N}-x\} + \min\{y, \sqrt{N}-y\} \\ &= \frac{2\sqrt{N}}{N-1} \sum_{x=1}^{\sqrt{N}-1} \min\{x, \sqrt{N}-x\}. \end{aligned}$$

The two cases follow from the last expression.

H Proof of Lemma 6

In the following we will only consider links l connecting two neighboring nodes, and paths $P = (l_1, l_2, \cdots, l_k)$ composed of such links. If $P = (l_1, l_2, \cdots, l_k)$, we say that the number of links on path P is k, and write $|P| = k$. A routing vector $H = (x_P : P \in \mathscr{P})$ is called a *shortest path* routing vector if $x_P > 0,\ P \in \mathscr{P}_{ij}$ implies $|P| = d(i, j)$.

In a network, for a given source destination pair there may exist many routes with the minimum path length. It is the objective of a routing vector to use shortest distances while distributing the load uniformly among links. In this appendix, we will prove that all *symmetric*, shortest path routing vectors satisfy (44). Before going into the details of what we mean by symmetry, it is useful to give an example. Consider a source-destination pair $i, j \in \mathcal{N}$. Define

$$\mathscr{P}'_{ij} = \{P \in \mathscr{P}_{ij} : |P| = d(i,j)\},$$

and

$$x_P = \begin{cases} \frac{1}{|\mathscr{P}'_{ij}|}, & \text{if } P \in \mathscr{P}'_{ij} \\ 0, & \text{otherwise.} \end{cases}$$

The vector $H = (x_P : P \in \mathscr{P})$ distributes routing load uniformly over all links with minimum path length. It is symmetric and satisfies (44).

We will use modulo arithmetic for discussing translation of nodes, links and paths. $(x, y) \in \mathcal{Z}^2$ refers to the node $(\text{mod}(x), \text{mod}(y))$ where $\text{mod}(x) = x \mod \sqrt{N}$ is the usual modulo function. When node $i = (x, y)$ is shifted by $\delta = (\delta(x), \delta(y))$, node

$$i + \delta = (x + \delta(x), y + \delta(y))$$

is obtained. Define the δ translation of link $l = (i, j)$ as $l + \delta = (i + \delta, j + \delta)$, and the δ translation of path $P = (l_1, l_2, \cdots, l_k)$ as

$$P + \delta = (l_1 + \delta, l_2 + \delta, \cdots, l_k + \delta).$$

We call a routing vector *shift invariant* if for all $i, j \in \mathcal{N}, P \in \mathscr{P}_{ij}, \delta \in \mathcal{Z}^2$,

$$x_P = x_{P+\delta}$$

is satisfied.

Denote the origin node with 0. Next, we will argue that if a routing vector is shift invariant then for every link l,

$$\sum_{i,j \in \mathcal{N}} \sum_{P \in \mathscr{P}_{ij}:\ l \in P} x_P = \sum_{j \in \mathcal{N}} \sum_{P \in \mathscr{P}_{0j}} x_P \psi(P, l), \tag{108}$$

where

$$\psi(P, l) = \sum_{i \in \mathcal{N}} 1(l \in P + i). \tag{109}$$

To see (108), make a change of variables $j' = j - i$,

$$\begin{aligned} \sum_{i,j \in \mathcal{N}} \sum_{P \in \mathscr{P}_{ij}} x_P 1(l \in P) &= \sum_{j' \in \mathcal{N}} \sum_{i \in \mathcal{N}} \sum_{P \in \mathscr{P}_{i,j'+i}} x_P 1(l \in P) \\ &= \sum_{j' \in \mathcal{N}} \sum_{P \in \mathscr{P}_{0,j'}} \sum_{i \in \mathcal{N}} x_{P+i} 1(l \in P + i) \\ &= \sum_{j' \in \mathcal{N}} \sum_{P \in \mathscr{P}_{0,j'}} x_P \sum_{i \in \mathcal{N}} 1(l \in P + i). \end{aligned}$$

Last equality is due to the translation invariance of the routing vector.

The links in a Manhattan network are in four directions: up,down,left,right. The number $\psi(P,l)$ is the number of links in P which are in the same direction with l. Definition (109) assures that $\psi(P,l)$ depends only the direction of l but not its location, *i.e.,* $\psi(P,l) = \psi(P,l+\delta)$ for every δ.

Equation (108) can be interpreted as follows. Suppose that the network traffic is uniform and equal to 1 for every source destination pair. Left hand side in (108) is the routing load over link l. Equation (108) implies that the routing load on each link l depends only on the direction of l. Moreover, (108) suggests an alternative view of computing routing load over link l: Fix the origin as the source node, and add up the traffic from the origin to the other nodes, passing through links in the same direction with l.

In order to define a symmetric routing vector, we need a few other definitions. Let the vertical reflection of node $i = (x,y)$ be $i_{\updownarrow} = (x,-y)$, the horizontal reflection be $i_{\leftrightarrow} = (-x,y)$, and the rotation be $i_{\circlearrowleft} = (-y,x)$. Similarly, let vertical reflection of a link $l = (i,j)$ be $l_{\updownarrow} = (i_{\updownarrow}, j_{\updownarrow})$, and vertical reflection of a path $P = (l_1, l_2, \cdots, l_k)$ be

$$P_{\updownarrow} = (l_{1\updownarrow}, l_{2\updownarrow}, \cdots, l_{k\updownarrow}).$$

Horizontal reflection and rotations of links $l_{\leftrightarrow}$, $l_{\circlearrowleft}$ and paths $P_{\leftrightarrow}$, $P_{\circlearrowleft}$ are defined similarly. Call a translation invariant routing vector *symmetric* if for all $P \in \mathscr{P}_{0j}$,

$$x_P = x_{P_{\updownarrow}} = x_{P_{\leftrightarrow}} = x_{P_{\circlearrowleft}} \tag{110}$$

is satisfied.

From the definition of Ψ, it follows that for all $* \in \{\updownarrow, \leftrightarrow, \circlearrowleft\}$,

$$\psi(P,l) = \psi(P_*, l_*). \tag{111}$$

Therefore, if a routing vector is symmetric then,

$$\begin{aligned} \sum_{j\in\mathcal{N}} \sum_{P\in\mathscr{P}_{0j}} x_P \psi(P,l) &= \sum_{j\in\mathcal{N}} \sum_{P\in\mathscr{P}_{0j}} x_{P_*} \psi(P_*, l_*) \\ &= \sum_{j\in\mathcal{N}} \sum_{P\in\mathscr{P}_{0j}} x_P \psi(P, l_*). \end{aligned} \tag{112}$$

The first equality follows from (110) and (111). The second equality is due to the fact that if we map every $P \in \mathscr{P}_{0j}$ to P_* then we again obtain the set $\mathscr{P}_{0j}$.

Let l be a link pointing up. Then, $l_{\circlearrowleft}, l_{\updownarrow}, (l_{\circlearrowleft})_{\leftrightarrow}$ are vectors pointing left,down and right, respectively. If the routing vector is symmetric, then (108) and (112) ensure that the traffic on links $l, l_{\circlearrowleft}, l_{\updownarrow}, (l_{\circlearrowleft})_{\leftrightarrow}$ are the same.

Let $H = (x_P : P \in \mathscr{P})$ be a symmetric, shortest path routing vector. Next, we will prove that H satisfies (44). From definitions, it follows that

$$|P| = \psi(P,l) + \psi(P,l_{\circlearrowleft}) + \psi(P,l_{\updownarrow}) + \psi(P,(l_{\circlearrowleft})_{\leftrightarrow}). \tag{113}$$

Therefore,

$$
\begin{aligned}
\sum_{i,j\in\mathcal{N}} \sum_{P\in\mathscr{P}_{ij}:\ l\in P} x_P &= \sum_{j\in\mathcal{N}} \sum_{P\in\mathscr{P}_{0j}} x_P \psi(P,l) \\
&= \frac{1}{4}\sum_{j\in\mathcal{N}} \sum_{P\in\mathscr{P}_{0j}} x_P(\psi(P,l)+\psi(P,l_{\circlearrowleft}) \\
&\qquad +\psi(P,l_{\updownarrow})+\psi(P,(l_{\circlearrowleft})_{\leftrightarrow})) \\
&= \frac{1}{4}\sum_{j\in\mathcal{N}} \sum_{P\in\mathscr{P}_{0j}} x_P|P| \\
&= \frac{1}{4}\sum_{j\in\mathcal{N}} d(0,j) \\
&= \frac{(N-1)\bar{L}}{4}. \qquad (114)
\end{aligned}
$$

The first equality is due to (108). The second in due to (112). The third one is due to (113). The fourth one is because $|P| = d(0,j)$ for each $P \in \mathscr{P}_{oj}$ and $\sum_{P\in\mathscr{P}_{0j}} x_P = 1$. The last equality follows from the definition of $\bar{L}$ and the symmetry of the network topology.

I Proof of the formula $\frac{i}{j}C_i$

Let node k receive packets from nodes $\{1,2,\cdots,j\}$. The packets from first i nodes $\{1,2,\cdots,i\}$ are for the receiver k. The rest of the packets are intended for other receivers, but node k happens to be in the neighborhood of each node in $\{i+1,i+2,\cdots,j\}$. The expected number of successful transmissions by the nodes $\{1,2,\cdots,i\}$ is

$$
\begin{aligned}
\mathbb{E}\{\sum_{r=1}^{i} 1(\text{node } r \text{ is successful})\} &= \sum_{r=1}^{i} \Pr\{\text{node } r \text{ is successful}\} \\
&= i\Pr\{\text{node 1 is successful}\} \\
&= i\tilde{C}_{j,1} \\
&= i\sum_{m=1}^{j} \binom{j-1}{m-1} \frac{C_{j,m}}{\binom{j}{m}} \\
&= i\sum_{m=1}^{j} \frac{m}{j} C_{j,m} \\
&= \frac{i}{j}C_j.
\end{aligned}
$$

J Converse to Theorem 5

Let λ be achievable with slotted ALOHA with retransmission probability q. Every packet delivered from node i to j must be successfully transmitted $d(i,j)$ times. Therefore,

$$\sum_{t=0}^{T-1} \sum_{i,j \in \mathcal{N}} W_{ij}(t) d(i,j) \leq \sum_{t=0}^{T-1} \sum_{l \in \mathcal{L}} 1(l \in \mathcal{F}(t)), \tag{115}$$

where the right hand side in the above equation is the total number of successful transmissions networkwide from slot zero to $T-1$. Observe the following chain of inequalities:

$$\begin{aligned}
N(N-1)\lambda \bar{L} &= \sum_{i,j \in \mathcal{N}} \lambda d(i,j) && (116)\\
&\leq \sum_{i,j \in \mathcal{N}} \mathbb{E}\{\liminf_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} W_{ij}(t)\} d(i,j) && (117)\\
&\leq \sum_{i,j \in \mathcal{N}} \liminf_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}\{W_{ij}(t)\} d(i,j) && (118)\\
&\leq \liminf_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} \sum_{i,j \in \mathcal{N}} \mathbb{E}\{W_{ij}(t)\} d(i,j) && (119)\\
&\leq \liminf_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} \sum_{l \in \mathcal{L}} \Pr(l \in \mathcal{F}(t)) && (120)\\
&= \sum_{l \in \mathcal{L}} \Pr(l \in \mathcal{F}(t)) && (121)\\
&= 4N \Pr(l \in \mathcal{F}(t)) && (122)\\
&= \frac{N}{4} \sum_{k=1}^{4} \binom{4}{k} q^k (1-q)^{5-k} C_k && (123)\\
&\leq N\bar{L}\, \eta_{ALOHA}. && (124)
\end{aligned}$$

Equation (116) is the definition of $\bar{L}$. Inequality (117) follows because λ is uniformly achievable (8). Inequality (118) holds because of Fatou's lemma. Inequality (120) can be seen by taking expectation in (115). Equality (122) is because there is total $4N$ (directed) links in a Manhattan network each with the same $\Pr(l \in \mathcal{F}(t))$. Equality (122) is because of (58). The converse follows.

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