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Optimal Operation and Economic Value of Energy Storage at Consumer Locations

Yunjian Xu and Lang Tong

***Abstract*—We study the optimal operation and economic value of energy storage operated by a consumer who seeks to maximize her long-term expected payoff (utility perceived from energy consumption minus energy cost). For a general setting that incorporates random electricity prices and the inter-temporal substitution effect in energy demand, we establish a threshold structure for optimal storage operation policies through a dynamic programming approach. For an important special case with inelastic energy demand, we prove that the consumer's maximum expected payoff is piecewise linear in the storage level; under an additional assumption that both the demand and prices are deterministic, we further establish the equivalence between the optimal storage operation problem and a minimum cost flow problem. These results significantly simplify the (exact) computation of optimal threshold policies. We define the value of storage (VoS) as the consumer's net benefit obtained by optimally operating the storage. We show that if the consumer can always buy and sell electricity at the same (realized) price, then it is optimal for her to use the storage only for arbitrage, and therefore the VoS does not depend on the consumer's demand.**

***Index Terms*—Energy storage, Value of storage, Demand response, Dynamic programming, Inter-temporal demand**

I. Introduction

Renewable generation capacity is expanding rapidly to potentially reduce carbon dioxide emissions and dependence on fossil fuels. As non-dispatchable generation, renewable energy introduces variability into the energy portfolio, and further amplifies the difficulty of matching demand with supply in the real time. Energy storage devices (e.g., batteries, flywheels, and plug-in electric vehicles) are environmentally friendly candidates that can provide flexibility to the system and mitigate the impact of volatile renewable generations [2].

The focus of this paper is on the operation of consumer-owned electric storages. Our motivation stems from the potential of electricity consumers to own and use storage devices (e.g., major consumers like data centers [3], [4] and individual consumers who own PHEVs (plug-in hybrid electric vehicles) [20], [5]) and from a recent study that shows consumer ownership of storage can be socially beneficial [6].

The operation of energy storage devices has received a lot of recent attention. The literature can be broadly divided into two categories: (i) articles that assume all parameters are deterministic time-varying quantities and make one-shot decisions such as how much to invest in renewable sources and energy storage, the optimal sizing of storage devices, etc., and (ii) articles that consider (either independent and identically distributed or Markovian) random demand, supply and/or costs, and study the operation of energy storage in a dynamic setting. The present paper belongs to the second category.

For category (i), the authors of [8] develop a nonlinear optimization program to evaluate the value of hydropower storage in offsetting wind power intermittence. The authors of [9] characterize the energy storage requirements for a balancing area or interconnections under different level of renewable penetration. The (marginal) value of energy storage devices is assessed in [10], [11], for two different settings with and without renewable generation.

We now turn our attention to existing works in category (ii). There exists a substantial literature on the operation of energy storage owned by renewable generators or system operators. The scheduling of energy storage systems has been studied in order to maximize the joint profit of wind farms and energy storage systems, through two-stage stochastic programming [12], stochastic optimal control [7], and model predictive control (MPC) [13]. The authors of [14] derive an upper bound on the marginal value of storage (at small installed capacities) for a transmission-constrained power network. A few recent works study the optimal operation of energy storage devices, in order to minimize the mismatch between the available renewable generation and system load [15], [16], [17]. We note that the aforementioned works focus on scheduling objectives that are different from our's; further, energy storage is assumed to have 100% charging and discharging efficiency in [15], and energy demand is assumed to be inelastic in [16], [17].

Another well studied application of energy storage is its use for arbitrage [19], [20], [21]. A few recent works take a dynamic programming approach to derive the arbitrage value of electric storage, in the presence of dynamic pricing [22], [23], [24], [25]. Different from the setting in the present paper, the aforementioned literature assumes that the operator of electric storages (e.g., an arbitrager) has no demand for electricity and puts no value on its own electricity consumption.

Closer to the present paper, there is a growing literature on the operation of consumer-owned electric storages. The scheduling of energy storage devices in smart homes has been studied through noncooperative game theoretic analysis [26], [27] and mixed integer quadratic programming [28]. Leveraging on techniques from Lyapunov optimization, a few recent papers propose a variety of on-line algorithms that

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are shown to be asymptotically optimal, as storage capacity increases to infinity [29]. These algorithms are expected to perform well when storage capacity is significantly larger than the maximum charging/discharging rates (i.e., if it takes many hours to fully charge and discharge the storage) [30].

Closely related to this work, the authors of [31], [32] study the optimal storage operation problem through a stochastic dynamic programming formulation, and prove the optimality of two-threshold policies similar to the one characterized in this paper. Our model (to be presented in the next section) extends the ones studied in [31], [32] in two significant ways.

1) While it is assumed that consumers cannot sell stored energy (back to the grid) in [31], and that consumers can always buy and sell at the same (realized) price in [32], our model allows the consumer to sell the stored energy (back to the grid) at an arbitrary price that is not higher than the (realized) purchasing price. We note that this generalization leads to non-trivial complication in the consumer's sequential decision making problem, due to the additional dimension of action the consumer could take to sell stored energy for profits. The mathematical framework constructed in this paper can be used to study the operation of storage owned by arbitrageurs, by setting the consumer's utility function to be constantly zero.
2) The consumer's demand is assumed to be inelastic in [32], and is assumed to be a random quantity that depends only on the current electricity (purchasing) price in [31]. The present work employs a more practical (and naturally, more complicated) consumer demand model that incorporates the *inter-temporal substitution effect* in energy demand. The inter-temporal substitution effect commonly exists in consumer demand, when a consumer maximizes her total utility by allocating resources (e.g., money) across time [33]. In the context of electricity consumption, the inter-temporal substitution effect exists in shiftable and deferrable loads [34], such as the charging of plug-in electric vehicles, dish washers, and clothes washers. For such appliances, it is often feasible to substitute energy consumption inter-temporally.
In our setting, the consumer has a (random) utility function at each stage, and seeks to maximize her long-term expected payoff, which is the sum of her total utility and the (possibly negative) net profit. Further, our model allows the consumer utility (perceived from energy consumption) to be inter-temporally coupled, *i.e.*, the energy consumption at stage t could influence the consumer's utility functions in future stages. It is worth noting that the incorporation of inter-temporally correlated consumer demand results in significant technical challenges in the characterization of optimal storage operation policies.[1]

In this paper, we construct a dynamic programming (D-P) framework to study the challenging sequential decision making problem on storage operation. For a general setting that incorporates both random electricity prices and inter-temporally correlated consumer demand, we characterize an optimal policy by two thresholds: (i) if the storage level is between the two thresholds, do not charge or discharge the storage; (ii) when the storage level is below the lower threshold, charge the storage; (iii) if the storage level is above the higher threshold, discharge the storage to meet the demand or/and to sell back to the market. To our knowledge, this work is the first that investigates the optimal storage operation problem in the context of inter-temporally correlated energy demand. For a special case with random but inter-temporally uncorrelated consumer utility functions, we provide a sharper characterization on an optimal two-threshold policy that can be effectively computed through backward induction.

The constructed DP framework enables us to study the value of storage (VoS), which is defined as the consumer's net benefit obtained by optimally operating the storage. Leveraging on the characterization of an optimal two-threshold policy, we show that the VoS is a concave function of storage capacity,[2] and can be computed by solving a sequence of (deterministic) convex optimization problems.

Computing optimal operation thresholds is in general computationally challenging. To this end, we consider an important special case where the consumer has (possibly random) *inelastic demand*,[3] which is the setting of several closely related works [16], [32]. We show that the consumer's maximum expected payoff (the optimal payoff-to-go) is piecewise linear in the storage level. This result significantly simplifies the (exact) computation of the characterized optimal threshold policy. Under an additional assumption that both energy demand and electricity prices are deterministic, we show that the storage operation problem is equivalent to a minimum cost flow problem. This equivalence result enables the application of a variety of minimum cost flow algorithms that can effectively solve the optimal storage operation problem, e.g., an optimal greedy algorithm with linear complexity in the number of stages (originally proposed in [39]).

For a practical setting where the consumer can always buy and sell electricity at the same (realized) price, we show that it is optimal for the consumer to use the storage only for arbitrage, by establishing the existence of an optimal policy that *never withdraws energy from the storage for consumption*. This result holds in a very general setting with arbitrary inter-temporally correlated consumer demand (cf. the discussion in Section V). This result enables the consumer to compute the optimal operation of energy storage without taking into account her (possibly complicated) energy demand.

We finally note that the storage operation problem is intimately related to inventory control problems with random production cost and uncertain demand [40], [41]. The optimality of similar threshold policies (e.g., the well known (s, S) policy) is established for inventory models with fluctuating demand [42], [43]. Our model differs from the classical

[1]For example, in our setting, the energy consumption at each stage is a complicated decision that depends on current electricity prices, the current storage level, future electricity prices, as well as the consumer's energy consumption history.

[2]We note that a qualitatively similar result is developed in [7], for a different setting with an objective to improve the expected profit of wind power producers.

[3]Electricity consumption usually exhibits inelasticity in the short term [35], [36], especially for large commercial and industrial consumers.

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setting of the inventory control literature in the following two aspects: (i) instead of the inventory holding cost that is proportional to the inventory level (storage level in our model), major losses resulting from storage operation are due to energy injection and withdrawal (e.g., battery charging and discharging), and (ii) while inventory could be used only to meet the (inelastic) demand, and unsatisfied demand is either backlogged or completely lost, our model allows the consumer to meet the demand (through purchasing energy from the grid) at a random per-unit cost, and to sell stored energy back to the grid at a random per-unit price. As a result, the optimal two-threshold policy characterized in this paper is different from (indeed, more complex than) (s, S)-type policies.[4]

The rest of the paper is organized as follows. Problem formulation is given in Section II. In Section III, we establish characterizations on optimal threshold policies. In Section IV, we consider an important special case where the consumer is faced with inelastic demand. In Section V, we formally define the value of storage and characterize some of its properties. In Section VI, we present numerical results using real pricing data from the New England ISO. We compute the value of storage under critical peak pricing and day-ahead hourly pricing, and benchmark a certainty-equivalent heuristic policy against the optimal policy under random prices. Section VII closes this paper with brief concluding remarks and possible directions for future work.

II. Model Formulation

We study the operation of a finite-capacity storage owned by an electricity consumer. The consumer has the options of discharging the storage for its own consumption, charging its battery from purchased power, and sell its stored electricity back to the grid. The detailed model of this decision process has the following elements:

1) **Discrete time**: Time periods are indexed by $t = 0, \ldots, T$.
2) **Storage capacity**: At each stage $t = 0, \ldots, T$, let $x_t \in [0, B]$ denote the storage level at the beginning of stage t, where $B > 0$ is the storage capacity (in kWh or MWh).
3) **Randomness**: For each stage $t = 0, \ldots, T$, let $s_t \in \mathcal{S}_t$ denote the global state. We assume that the set $\mathcal{S}_t$ is finite. The global state evolves as an exogenous Markov chain, of which the transition probability is independent of the consumer's action. The global state contains all the currently available information about all exogenous factors that have impacts on the consumer's current/future demand and payment, such as the current electricity prices, predictions on future (possibly random) electricity prices, and weather conditions.
4) **Prices**: For $t = 0, \ldots, T-1$ and every global state $s_t \in \mathcal{S}_t$, let $p_t(s_t) \in (0, \infty)$ and $q_t(s_t) \in [0, p_t(s_t)]$ denote the electricity purchasing price and the selling price (from the grid), respectively (in \$/kWh or \$/MWh).
5) **Electricity consumption**: For $t = 0, \ldots, T-1$, let $c_t \in [0, \infty)$ denote the amount of energy purchased (from the grid) for consumption at stage t, and let $d_t \in [0, \infty)$ be the amount of energy withdrawn from the storage for consumption.
6) **Consumer state**: We let $f_t \in \mathbb{R}$ denote the state of consumer, which reflects the consumer's desirability for energy consumption, and has an impact on her utility perceived from electricity consumption.
7) **Consumer utility**: For $t = 0, \ldots, T-1$, the consumer receives a utility $u_t(c_t + d_t, s_t, f_t)$, which depends on her electricity consumption $c_t + d_t$, the global state s_t, and her own state f_t. Naturally, the utility function is non-decreasing in her desirability for energy consumption f_t. For every $t \in \{0, \ldots, T-1\}$ and every $s_t \in \mathcal{S}_t$, the utility function is concave in $(c_t + d_t, f_t)$ and non-decreasing and continuously differentiable in $c_t + d_t$, with[5]
$$\lim_{y \to \infty} u_t'(y, s_t, f_t) = 0, \qquad \forall\, f_t \in \mathbb{R}. \tag{1}$$
8) **Charging and discharging**: For $t = 0, \ldots, T-1$, the consumer purchases $a_t \in \mathbb{R}$ amount of electricity for storage. a_t is negative if the consumer sells the stored energy to the market. For the rest of the paper, we use a_t^+ and a_t^- to denote the positive and negative part of a_t, respectively, i.e., $a_t^+ = \max\{0, a_t\}$ and $a_t^- = -\min\{0, a_t\}$.
9) **Rate constraints**: Let R^C and R^D denote the maximum storage charging and discharging rate within one time period. We have
$$\gamma a_t^+ \le R^C, \qquad (d_t + a_t^-)/\eta \le R^D. \tag{2}$$
10) **Evolution of storage level**: Given the storage level and the action taken at stage t, the storage level at the next stage is
$$x_{t+1} = x_t + \gamma a_t^+ - (d_t + a_t^-)/\eta, \tag{3}$$
where $\gamma \in (0, 1]$ is the charging efficiency and $\eta \in (0, 1]$ is the discharging efficiency.[6] We assume that the storage level at the initial stage 0, x_0, is exogenous and independent of the consumer's decision.
11) **Evolution of consumer state**: We assume that the transition of the consumer's state is affine, i.e.,
$$f_{t+1} = \alpha_t f_t + \beta_t (c_t + d_t) + \vartheta_t(s_t), \qquad t = 0, \ldots, T-1, \tag{4}$$
where $\alpha_t \in [0, \infty)$ and $\beta_t \in [-1, 0]$ are given real numbers, and $\vartheta_t : \mathcal{S}_t \to \mathbb{R}$ is an arbitrary mapping. β_t is assumed to be non-positive because the consumer's desirability for energy consumption at stage $t+1$, f_{t+1}, is non-increasing in her energy consumption $c_t + d_t$.

We note that the linear dynamic system formulated above is more general than the settings of the most closely related liter-

[4]Under an (s, S)-type policy, if the inventory level is below the threshold s then increase the inventory to the order-up-to level S [42], [43]. The two-threshold policy characterized in this paper, on the other hand, has an additional threshold if the storage level is above which then discharge the storage for consumption or selling back to the grid.

[5]Here, and in what follows, we use u_t' to denote the derivative of u_t with respect to its first argument (the total energy consumption at stage t).

[6]Note that we have omitted the self-discharge of the energy storage device, since for most batteries their self-discharge rate (typically a few percent per month) is negligible compared to charging/discharging inefficiencies. Indeed, the storage efficiency of many different types of modern batteries (e.g., Lead acid, Li-ion, and Vanadium redox batteries) is close to 100% [44].

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ature (that applies dynamic programming to study consumer-owned energy storage operation). We elaborate below.

1) A natural special case of our model is the setting with a single consumer state (i.e., $\alpha_t = 1$, $\beta_t = 0$, and $\vartheta_t(s_t) = 0$ for every t). In this setting, the consumer has random utility functions that are not inter-temporally correlated. This setting allows the consumer to tradeoff her energy consumption with the current electricity price as well as the value of storage (that depends on future prices and demand), and is indeed more general than the models studied in related works [31], [32].
2) The introduction of consumer state f_t enables us to model the inter-temporal substitution effect in consumer demand: the action taken at stage t can influence the utility function at stage $t+1$ through the state f_{t+1}. Our setting with a time-varying consumer state f_t can therefore incorporate a larger category of consumer demands. As an example, consider appliances that require a certain amount of energy consumption before a certain time, e.g., deferrable loads such as the charging of PHEVs, dish washers, and clothes washers [48], [49]. The consumer state f_t can be set as the remaining amount of energy needed to complete the task (with $\alpha_t = 1$ and $\beta_t = -1$ for every t). The consumer utility function becomes constantly zero if the consumer state f_t becomes zero. Our formulation allows the consumer utility to depend on the remaining amount of energy needed to finish the task (through the consumer state f_t).

We are now ready to formulate the operation problem as a $(T+1)$-stage dynamic program (DP) by introducing its state space, action sets, transition probabilities, and stage cost. At each stage t, the **system state** consists of the current storage level, x_t, the global state s_t, and the consumer state f_t. For notational convenience, we let $\mathbf{z}_t = (x_t, s_t, f_t)$ denote the system state at stage t.

For $t = 0, \dots, T-1$, a feasible **action** is a vector (a_t, c_t, d_t) that satisfies the rate constraints in (2), and the following constraint:

$$0 \le x_t + \gamma a_t^+ - (d_t + a_t^-)/\eta \le B, \tag{5}$$

where the first inequality is to avoid storage underflow, and the second inequality is to prevent storage overflow. We let $\mathcal{A}(x_t)$ denote the (convex and compact) set of feasible actions at storage level x_t. Without loss of generality, we have assumed (in (5)) that the consumer never charges and withdraws the storage simultaneously, i.e., $d_t > 0$ implies $a_t \le 0$ and $a_t > 0$ implies $d_t = 0$.

The evolution of storage levels is deterministic, and is governed by (3). The evolution of the global state is random and independent of the current storage level and the actions taken by the consumer. For $t = 0, \dots, T-1$, the **stage payoff** is given by

$$\begin{aligned} &w_t(\mathbf{z}_t, a_t, c_t, d_t) \\ &= u_t(c_t + d_t, s_t, f_t) - p_t(s_t)(a_t^+ + c_t) + q_t(s_t)a_t^-. \end{aligned} \tag{6}$$

At the terminal stage T, no action is available, and the stage payoff is given by $w_T(x_T, s_T, f_T)$, which reflects the salvage value of storage. Suppose that $w_T(x_T, s_T, f_T)$ is continuously differentiable in x_T, non-decreasing and concave in (x_T, f_T) for every $s_T \in \mathcal{S}_T$. We further assume that the cross derivatives of the terminal-stage payoff function are non-negative, i.e., under any global state s_T, $w_T(x_T', s_T, f_T) - w_T(x_T, s_T, f_T)$ is non-decreasing in f_T, for any given $x_T' > x_T$, and that $w_T(x_T, s_T, f_T') - w_T(x_T, s_T, f_T)$ is non-decreasing in x_T, for any given $f_T' > f_T$.[7]

A **policy** $\pi = (\mu_0, \dots, \mu_{T-1})$ is a sequence of decision rules such that $\mu_t(\mathbf{z}_t) \in \mathcal{A}(x_t)$, for all $\mathbf{z}_t$ and t. We let $V_t^\pi(\mathbf{z}_t)$ denote the payoff-to-go function under a policy π and the current system state $\mathbf{z}_t = (x_t, s_t, f_t)$:

$$\begin{aligned} V_t^\pi(\mathbf{z}_t) &= w_t(\mathbf{z}_t, \mu_t(\mathbf{z}_t)) \\ &+ \mathbb{E}\left\{\textstyle\sum_{\tau=t+1}^{T-1} w_\tau(\mathbf{z}_\tau, \mu_\tau(\mathbf{z}_\tau)) + w_T(\mathbf{z}_T) \mid s_t\right\}, \end{aligned} \tag{7}$$

where the expectation is over the sequence of global states $\{s_\tau\}_{\tau=t+1}^T$, conditioned on the current global state s_t. We note that since the marginal utility converges to zero (cf. (1)), and the purchasing price is always positive, the payoff-to-go function is always bounded, under any policy π. By a slight abuse of notation, we use $V_t(\mathbf{z}_t)$ to denote the optimal payoff-to-go function, i.e.,

$$V_t(\mathbf{z}_t) \triangleq \sup_\pi \{V_t^\pi(\mathbf{z}_t)\}. \tag{8}$$

We say a policy π^* is optimal, if it attains the optimal payoff-to-go defined above, i.e., $V_0^{\pi^*}(\mathbf{z}_0) = V_0(\mathbf{z}_0)$, for all initial states $\mathbf{z}_0$.

III. Optimal Storage Operation

In this section, we study the storage operation problem faced by the consumer. In Section III-A, we first consider the general formulation introduced in Section II. We show that the optimal payoff-to-go is concave in (x_t, f_t), which enables us to establish a threshold structure of an optimal policy in Theorem 3.1. In Section III-B, we consider a special case without inter-temporally correlated demand, and provide a sharper (threshold) characterization on an optimal policy.

A. The General Case

Lemma *3.1:* For $t = 0, \dots, T$ and every $s \in \mathcal{S}_t$, the optimal payoff-to-go function $V_t(x, s, f)$ is concave in the vector (x, f). □

The proof of Lemma 3.1 is given in Appendix A. For $t = 0, \dots, T-1$, the Bellman's equation yields

$$\begin{aligned} V_t(x_t, s_t, f_t) = \max_{(a_t, c_t, d_t)} &\{w_t(x_t, s_t, f_t, a_t, c_t, d_t) \\ &+ \bar{V}_{t+1|s_t}(x_{t+1}, f_{t+1})\}, \end{aligned} \tag{9}$$

where x_{t+1} is determined by (3), f_{t+1} is given by (4), and $\bar{V}_{t+1|s_t}(x_{t+1}, f_{t+1})$ denote the (conditional) expected payoff-to-go function at stage $t+1$, i.e., for $t = 0, \dots T-1$

$$\bar{V}_{t+1|s_t}(x_{t+1}, f_{t+1}) \triangleq \mathbb{E}\{V_{t+1}(x_{t+1}, s_{t+1}, f_{t+1}) \mid s_t\}, \tag{10}$$

[7] This assumption naturally holds for deferrable loads with f_T denoting the remaining amount of energy needed to complete the task.

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where the expectation is over the global state s_{t+1}, provided that the global state at stage t is s_t. Since the feasible action space $\mathcal{A}(x_t)$ is convex, it is straightforward to check that given the optimal payoff-to-go at stage $t+1$, the optimization problem on the right hand side of (9) is concave. Later in this section, we will provide (in Theorem 3.1) a characterization on an optimal policy that solves the optimization problem on the right hand side of (9).

It follows from Lemma 3.1 and (10) that $\bar{V}_{t|s}(x, f)$ must be concave, for every t and $s \in \mathcal{S}_t$. We therefore have the existence of its right and left directional derivatives on both x and f.[8] For the rest of this paper, we use notations $\partial_x^+ \bar{V}_{t|s}(x, f)$ and $\partial_x^- \bar{V}_{t|s}(x, f)$ to denote the right and left directional derivative of $\bar{V}_{t|s}(x, f)$ with respect to x, respectively. $\partial_f^+ \bar{V}_{t|s}(x, f)$ and $\partial_f^- \bar{V}_{t|s}(x, f)$ are similarly defined.

We write the expected payoff-to-go on the right hand side of (9) in terms of the action vector (a_t, c_t, d_t),

$$\begin{aligned} &u_t(c_t + d_t, s_t, f_t) - p_t(s_t)(a_t^+ + c_t) + q_t(s_t)a_t^- \\ &\qquad + \bar{V}_{t+1|s_t}\left(x_t + \gamma a_t^+ - (d_t + a_t^-)/\eta, f_{t+1}\right), \end{aligned} \tag{11}$$

where $f_{t+1} = \alpha_t f_t + \beta_t(c_t + d_t) + \vartheta_t(s_t)$.

Eq. (11) enables us to write the first-order conditions that are necessary and sufficient for an action vector (a_t, c_t, d_t) to be optimal under a given system state (x_t, s_t, f_t); these first-order conditions are given in Eq. (12) (locating at the top of next page), where x_{t+1} is given by (3) and f_{t+1} is given by (4). Given the current system state $\mathbf{z}_t = (x_t, s_t, f_t)$, we let $y_t(\mathbf{z}_t)$ denote the maximum optimal amount of energy procurement for consumption if a_t and d_t are forced to be zero, i.e., $y_t(\mathbf{z}_t)$ is the maximum c_t that satisfies the first two conditions in (12), with $d_t = 0$, $x_{t+1} = x_t$, and $f_{t+1} = \alpha_t f_t + \beta_t c_t + \vartheta_t(s_t)$.

The following lemma establishes conditions under which it is optimal not to charge or discharge the storage.

Lemma *3.2:* Given the current system state $\mathbf{z}_t$, it is optimal not to charge or discharge the storage, i.e., $a_t^* = 0$ and $d_t^* = 0$, if the following two conditions hold simultaneously

$$\begin{aligned} \max\left\{q_t(s_t), u_t'(y_t(\mathbf{z}_t), s_t, f_t) + \beta_t \partial_f^- \bar{V}_{t+1|s}(x_t, f_{t+1})\right\} \\ -\frac{1}{\eta}\partial_x^- \bar{V}_{t+1|s_t}(x_t, f_{t+1}) \le 0, \end{aligned} \tag{13}$$

and

$$\gamma \partial_x^+ \bar{V}_{t+1|s_t}(x_t, f_{t+1}) - p_t(s_t) \le 0, \tag{14}$$

where $f_{t+1} = \alpha_t f_t + \beta_t y_t(\mathbf{z}_t) + \vartheta_t(s_t)$. □

Lemma 3.2 directly follows from the first-order conditions in (12). We now define the two thresholds that characterize an optimal operation policy in Theorem 3.1. We let $k_t(s_t, f_t)$ denote the maximum x_t in the interval $[0, B]$ such that the condition in (13) holds; if the condition in (13) does not hold even for $x_t = 0$, we let $k_t(s_t, f_t) = 0$. We let $h_t(s_t, f_t)$ denote the minimum x_t in the interval $[0, B]$ such that the condition in (14) holds; if the condition in (14) does not hold even for $x_t = B$, we let $h_t(s_t, f_t) = B$.

It is straightforward to check (from the first-order conditions in (12)) that the two conditions in Eqs. (13) (14) cannot be violated simultaneously, because given any system state $\mathbf{z}_t = (x_t, s_t, f_t)$,

$$u_t'(y_t(\mathbf{z}_t), s_t, f_t) + \beta_t \partial_f^- \bar{V}_{t+1|s}(x_t, f_{t+1}) \le p_t(s_t). \tag{15}$$

It follows that for any $x_t > k_t(s_t, f_t)$, since the condition in (13) is violated, the condition in (14) must hold. We therefore have $h_t(s_t, f_t) \le k_t(s_t, f_t)$, for all possible s_t and f_t.

Theorem *3.1:* There exists an optimal policy $\pi^* = (\mu_0^*, \dots, \mu_{T-1}^*)$ characterized by the two thresholds $h_t(s_t, f_t)$ and $k_t(s_t, f_t)$.

(a) If the storage level x_t is below the threshold $h_t(s_t, f_t)$, then charge the storage and purchase electricity for consumption, i.e., $d_t^* = 0$, $a_t^* > 0$, and $c_t^* \ge 0$.

(b) If x_t is between the two thresholds, i.e., if $x_t \in [h_t(s_t, f_t), k_t(s_t, f_t)]$, then keep the storage level, and purchase electricity for consumption, i.e., $a_t^* = 0$, $c_t^* = y_t(\mathbf{z}_t)$, and $d_t^* = 0$.

(c) If x_t is above the threshold $k_t(s_t, f_t)$, then discharge the storage to meet demand and/or sell back to the market, i.e., $a_t^* \le 0$, $c_t^* \ge 0$ and $d_t^* \ge 0$. □

The proof of Theorem 3.1 relies on the concavity of the value function $\bar{V}_{t+1|s_t}$ and the first-order conditions in (12). This proof is deferred to Appendix B.

B. A Special Case without Inter-temporally Correlated Demand

In this subsection, we consider a special case with a single consumer state, i.e., $\alpha_t = 1$, $\beta_t = 0$, and $\vartheta_t(s_t) = 0$ for every t. Since there is a single consumer state, we will drop f_t from all notations used in this subsection.

In this special case, the two thresholds no longer depend on $y_t(\mathbf{z}_t)$. The threshold $k_t(s_t)$ is defined as the maximum x_t in the interval $[0, B]$ such that the condition in (13) holds,

$$\partial_x^- \bar{V}_{t+1|s_t}(x_t) \ge \eta \max\{q_t(s_t), \min\{p_t(s_t), u_t'(0, s_t)\}\}.$$

We note that $k_t(s_t)$ is simply the threshold $k_t(s_t, f_t)$ (defined in Section III-A) for the special case with a single consumer state. In this case, the term $u_t'(y_t(\mathbf{z}_t), s_t, f_t)$ in (13) equals $\min\{p_t(s_t), u_t'(0, s_t)\}$, because at the optimal procurement level $y_t(\mathbf{z}_t)$ the marginal consumer utility equals $p_t(s_t)$ as long as $y_t(\mathbf{z}_t) > 0$.

Let $h_t(s_t)$ denote the minimum x_t in the interval $[0, B]$ such that the following condition holds,

$$\gamma \partial_x^+ \bar{V}_{t+1|s_t}(x_t) \le p_t(s_t).$$

A sharper characterization of an optimal two-threshold policy is provided in Corollary 3.1. We first (re-)define a few notations that would be useful in this corollary. Analogous to $y_t(\mathbf{z}_t)$ defined in Section III-A, the scalar $y_t(s_t)$ is defined as the optimal amount of energy procurement (for consumption) if a_t and d_t are set to be zero:

$$y_t(s_t) \triangleq \begin{cases} \max\{y : u_t'(y, s_t) = p_t(s_t)\}, & \\ & \text{if} \quad u_t'(0, s_t) > p_t(s_t), \\ 0, & \text{otherwise}. \end{cases} \tag{16}$$

[8] Indeed, since $\bar{V}_{t|s}(\cdot)$ is concave, it must be continuously differentiable at all points in $(0, B) \times \mathbb{R}$ but at most countably many points.

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$$
\begin{cases}
u_t'(c_t+d_t,s_t,f_t)-p_t(s_t)+\beta_t\partial_f^-\bar{V}_{t+1|s_t}(x_{t+1},f_{t+1})\le 0, & \text{if } c_t\ge 0,\\
u_t'(c_t+d_t,s_t,f_t)-p_t(s_t)+\beta_t\partial_f^+\bar{V}_{t+1|s_t}(x_{t+1},f_{t+1})\ge 0, & \text{if } c_t> 0,\\
u_t'(c_t+d_t,s_t,f_t)-\dfrac{1}{\eta}\partial_x^-\bar{V}_{t+1|s_t}(x_{t+1},f_{t+1})+\beta_t\partial_f^-\bar{V}_{t+1|s_t}(x_{t+1},f_{t+1})\le 0, & \text{if } d_t\ge 0,\\
u_t'(c_t+d_t,s_t,f_t)-\dfrac{1}{\eta}\partial_x^+\bar{V}_{t+1|s_t}(x_{t+1},f_{t+1})+\beta_t\partial_f^+\bar{V}_{t+1|s_t}(x_{t+1},f_{t+1})\ge 0, & \text{if } d_t> 0,\\
-p_t(s_t)+\gamma\partial_x^+\bar{V}_{t+1|s_t}(x_{t+1},f_{t+1})\le 0, & \text{if } a_t^+\ge 0,\\
-p_t(s_t)+\gamma\partial_x^-\bar{V}_{t+1|s_t}(x_{t+1},f_{t+1})\ge 0, & \text{if } a_t^+> 0,\\
q_t(s_t)-\dfrac{1}{\eta}\partial_x^-\bar{V}_{t+1|s_t}(x_{t+1},f_{t+1})\le 0, & \text{if } a_t^-\ge 0,\\
q_t(s_t)-\dfrac{1}{\eta}\partial_x^+\bar{V}_{t+1|s_t}(x_{t+1},f_{t+1})\ge 0, & \text{if } a_t^-> 0,
\end{cases}
\tag{12}
$$

$y_t(s_t)$ is indeed the maximum optimizer that maximizes the stage payoff of a consumer without storage, $u_t(y,s_t)-yp_t(s_t)$, over $y\in[0,\infty)$. Analogously, we define $e_t(s_t)$ as the maximum optimizer that maximizes $u_t(z,s_t)-eq_t(s_t)$.

Finally, we let $g_t(s_t)\in[0,B]$ denote the maximum storage level at which the expected marginal storage value (at stage $t+1$) is no less than the adjusted selling price $\eta q_t(s_t)$, i.e.,

$$
g_t(s_t)\triangleq
\begin{cases}
0, & \text{if } \partial_x^+\bar{V}_{t+1|s}(0)\le\eta q_t(s_t),\\
B, & \text{if } \eta q_t(s_t)\le\partial_x^-\bar{V}_{t+1|s}(B),\\
\max\{g\in[0,B):\partial_x^+\bar{V}_{t+1|s}(g)\ge\eta q_t(s_t)\}, & \text{otherwise.}
\end{cases}
\tag{17}
$$

The concavity of $\bar{V}_{t+1|s}(\cdot)$ and the fact that $q_t(s)\le p_t(s)$ imply that $g_t(s)\ge h_t(s)$. We therefore have $h_t(s_t)\le k_t(s_t)\le g_t(s_t)$.

Corollary *3.1:* Suppose that there is a single consumer state, i.e., $\alpha_t=1$, $\beta_t=0$, and $\vartheta_t=0$ for every t. There exists an optimal policy $\pi^*=(\mu_0^*,\dots,\mu_{T-1}^*)$ characterized by the two thresholds $h_t(s_t)$ and $k_t(s_t)$.

(a) If the storage level x_t is below the threshold $h_t(s_t)$, then greedily charge the storage up to this level, and purchase electricity for consumption, i.e., $c_t^*=y_t(s_t)$, $d_t^*=0$, and

$$a_t^*=\min\left\{R^C/\gamma,(h_t(s_t)-x_t)/\gamma\right\},$$

where R^C is the maximum charging rate.

(b) If the storage level x_t is between the two thresholds, i.e., if $x_t\in[h_t(s_t),k_t(s_t)]$, then keep the storage level, and purchase electricity for consumption, i.e., $a_t^*=0$, $c_t^*=y_t(s_t)$, and $d_t^*=0$.

(c) If the storage level x_t is above the threshold $k_t(s_t,f_t)$, then greedily discharge the storage to meet demand and/or sell back to the market. d_t^* is given in (18) (locating at the top of next page),

$$
\begin{aligned}
(a_t^*)^-&=-\min\{a_t^*,0\}\\
&=\left(\eta\min\left\{x_t-g_t(s_t),R^D\right\}-d_t^*\right)^+,
\end{aligned}
$$

and $c_t^*=(y_t(s_t)-d_t^*)^+$. □

The proof of Corollary 3.1 is given in Appendix C. We note that for the special case with a single consumer state, Corollary 3.1 provides a sharper characterization (than Theorem 3.1) on an optimal two-threshold policy that can be computed in closed form by backward induction.

IV. Computation of the Optimal Threshold Policy under Inelastic Demand

In this section, we consider a special case of the model formulated in Section II, where each consumer is faced with inelastic energy demand. In Section IV-A, we prove that the optimal payoff-to-go function is piecewise linear in the storage level x_t, for a case with random (inelastic) demand and electricity prices. This important structural result could significantly simplify the computation of the optimal threshold policy characterized in Corollary 3.1. In Section IV-B, for the case with deterministic (inelastic) demand and electricity prices, we formulate the optimal storage operation as a minimum cost flow problem, and establish the optimality of a simple greedy algorithm with linear complexity in the number of stages.

Within this section, we assume that the salvage value is linear, i.e.,

$$V_T(x_T,s_T)=\eta q_T(s_T)x_T,\tag{19}$$

where $q_T\ge 0$ is a deterministic constant.

A. Stochastic Demand and Prices

For $t=0,\dots,T-1$, let $\ell_t(s_t)$ denote the consumer's inelastic demand at stage t, i.e., the consumer demands $\ell_t(s_t)$ amount of energy consumption at the global state s_t, regardless of the current and future electricity prices. Note that the inelastic demand setting can be viewed as a special case of utility-function based model formulated in Section II, by letting

$$u_t(c_t+d_t,s_t)=R\cdot\min\{c_t+d_t,\ell_t(s_t)\},\qquad t=0,\dots,T-1,\tag{20}$$

where c_t+d_t is the energy consumption at stage t, and the marginal utility R is larger than the highest possible purchasing price. Since there is only one consumer state, within this subsection we will drop f_t from notations.

$$d_t^* = \begin{cases} \sup_{d \le \min\{\eta x_t, e_t(s_t), \eta R^D\}} \left\{ d : \partial_x^+ \bar{V}_{t+1}(x_t - d/\eta) \le \eta \min\{u_t'(d, s_t), p_t(s_t)\} \right\}, & \text{if } u_t'(0, s_t) > q_t(s_t), \\ 0, & \text{otherwise}, \end{cases} \tag{18}$$

Theorem *4.1:* Suppose that the consumer's energy demand is inelastic (given in Eq. (20)). For $t = 0, \dots, T-1$, $\bar{V}_{t+1|s_t}(x)$ is non-decreasing, concave, and piecewise linear in x. As a result, $\partial^{\pm} \bar{V}_{t+1|s_t}(x)$ are step functions over $x \in [0, B]$.

The proof of Theorem 4.1 is given in Appendix D.

Remark *4.1:* The structural result established in Theorem 4.1 significantly simplifies the computation of the optimal policy characterized in Corollary 3.1. We first note that the computation of this optimal threshold policy relies on the computation of the partial derivatives $\{\partial^{\pm} \bar{V}_{t+1|s_t}(x_{t+1})\}$, for $t = 0, \dots, T-1$ and $s_t \in \mathcal{S}_t$. The computation of these partial derivatives over the continuous interval $x_{t+1} \in [0, B]$ is in general highly non-trivial. The structural result established in Theorem 4.1 ensures that these partial derivatives can be computed in finite time. Indeed, Eqs. (38) and (39) (in Appendix D) provide an efficient algorithm that computes these partial derivatives through backward induction.

B. Deterministic Demand and Prices

In this subsection, we consider the case with deterministic (inelastic) demand and energy price. Since there is only one global state, within this subsection we further drop s_t from notations. The setting of deterministic prices is motivated by the fact that many large commercial and industrial consumers are charged day-ahead prices that are released in advance [37], [38]. Under day-ahead hourly pricing, for a consumer who plans for daily storage operation, the hourly electricity prices (released one day ahead of the real time) can be regarded as deterministic.

For the special case considered in this subsection, the optimal storage operation can be obtained by solving a linear optimization problem. Since $c_t + d_t$ must equal the inelastic demand at stage t, there are essentially two decision variables at each stage. We note that the complexity of this linear program is roughly cubic in the number of stages.[9] We will formulate the storage operation problem as a minimum cost flow problem, which can be solved by a simple greedy algorithm with linear complexity in the number of stages.

We now introduce the minimum cost flow problem that is equivalent to the storage operation problem faced by the consumer. The graph presented in Fig. 1 has $2T + 2$ nodes and $3T$ directed arcs. The graph is connected in that there is a directed path between every pair of nodes. We let $\mathcal{N}$ and $\mathcal{L}$ denote the set of nodes and arcs, respectively. Each arc (i, j) is associated with a two dimensional vector (c_{ij}, u_{ij}) indicating the cost and the capacity of this arc. We also associate each node i with a real number $b(i)$ that represents demand at node i (supply if $b(i) < 0$):

- for the T nodes $\{n_t\}_{t=0}^{T-1}$, the demand $b(n_t)$ equals ℓ_t, which is the consumer's inelastic energy demand at stage t;
- for the "source" node s, its demand is $b(s) = -\sum_{t=0}^{T-1} \ell_t$;
- the demand of the other $T + 1$ nodes is zero.

A set of network flows $\{y_{ij}\}_{(i,j) \in \mathcal{L}}$ is **feasible** if it respects the capacity constraints of all arcs, and balances the demand and supply, i.e.,

$$y_{ij} \le u_{ij}, \quad \forall (i, j) \in \mathcal{L}; \quad \sum_j y_{ji} - \sum_j y_{ij} = b(i), \quad \forall i \in \mathcal{N}.$$

The objective of the minimum cost flow problem is to find a feasible set of flows that minimize the total cost, $\sum_{(i,j) \in \mathcal{L}} c_{ij} y_{ij}$.

We now argue that the formulated minimum cost flow problem is equivalent to the optimal storage operation problem. For $t = 0, \dots, T-1$, the flow on arc (s, m_t) is $\eta\gamma a_t^+$, the amount of "withdrawable" electricity charged into the storage at stage t. The flow on each arc (s, n_t) is c_t, the amount of electricity purchased at stage t for consumption. The flow on each arc (m_t, n_t) is $a_t^- + d_t$, the amount of energy withdrawn from the storage (for consumption and selling back to the grid) at stage t. The flow on each arc (m_{t-1}, m_t) is ηx_t, the amount of "withdrawable" electricity in the storage at the beginning of stage t. The flow on each arc (n_t, d) is a_t^-, the amount of energy sold to the grid at stage t. We note that the constraints in (2) and (5) are incorporated into the minimum cost flow problem by the capacity constraints of these arcs.

It is worth noting that whenever there is positive flow on an arc (m_t, n_t), i.e., if $a_t^- + d_t > 0$, then it is suboptimal to have positive flow on the arc (s, m_t), i.e., a_t^+ must be zero, because it is cheaper procure energy from the grid through the arc (s, n_t). For $t = 0, \dots, T-1$, the stored energy at node m_t can be either kept for future usage (through the arc (m_t, m_{t+1}) to node m_{t+1}) or withdrawn at stage t. For $t \le T - 1$, the energy withdrawn from the storage can be consumed through the arc (m_t, n_t) to node n_t, or sold back to the grid through the arcs (m_t, n_t) and (n_t, d) to the "destination" node d. For the terminal stage T, there is no limit on the amount of energy that could be "withdrawn" and therefore the arcs (m_T, n_T) and (n_T, d) have infinite capacity.

The cost resulting from a feasible set of network flows $\{y_{ij}\}_{(i,j) \in \mathcal{L}}$ is given by

$$\begin{aligned} &\sum_{t=0}^{T-1} \left(p_t y_{s,n_t} + \frac{p_t}{\eta\gamma} y_{s,m_t} \right) - \sum_{t=0}^{T} q_t y_{n_t,d} \\ &= \sum_{t=0}^{T-1} p_t \left(c_t + a_t^+ \right) - \sum_{t=0}^{T-1} q_t a_t^- - \eta q_T x_T, \end{aligned} \tag{21}$$

where the equality follows from the discussion above, i.e., $y_{s,n_t} = c_t$, $y_{s,m_t} = \eta\gamma a_t^+$, $y_{n_t,d}$ equals a_t^- for $t < T$ and ηx_T for $t = T$. We note that the left hand side of (21) is the total network flow cost, and the right hand side of (21) is the

[9] For example, for a storage owner who faces five-minute real-time balancing prices, the daily storage operation problem has 288 stages, and a weekly storage operation problem has 2016 stages.

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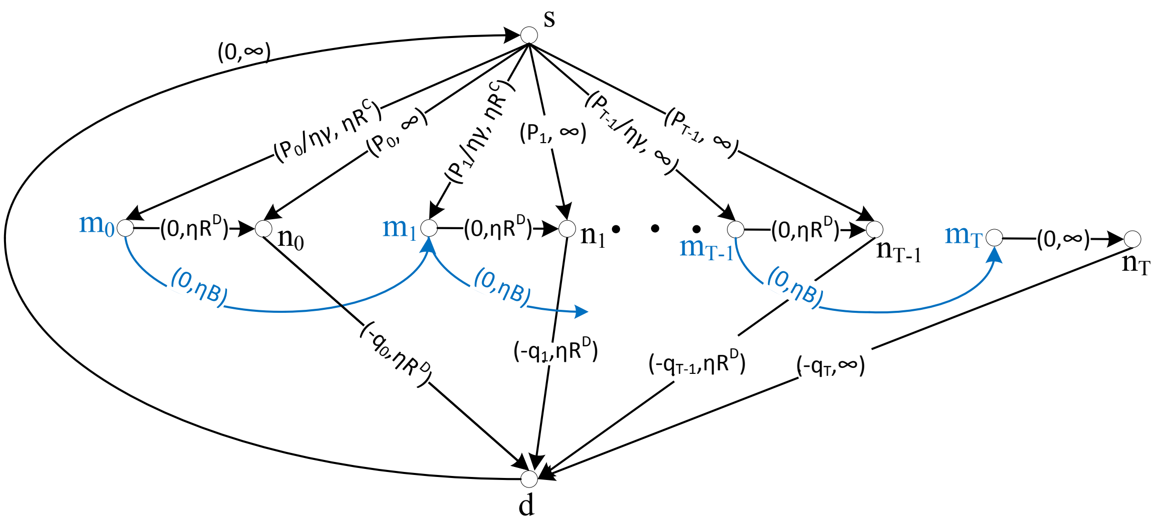

Fig. 1. An equivalent minimum cost flow problem.

consumer's total energy cost.

We have argued that (i) a feasible set of network flows corresponds to a feasible storage operation policy, and (ii) the total cost resulting from a feasible set of network flows equals the negative of the consumer's total payoff. As a result, the network flow cost minimization problem depicted in Fig. 1 is equivalent to the optimal storage operation problem.

As one of the most fundamental problems in network flow theory, the minimum cost flow problem has been extensively explored [45], [46]. Next we will introduce a widely used algorithm (referred to as right-hand-side (RHS) scaling algorithm in this paper) originally proposed in [39], and then discuss its optimality and complexity.

The RHS scaling algorithm was proposed to solve uncapacitated minimum cost flow problems. We therefore first apply a well-known transformation to convert the capacitated minimum cost flow problem (defined in Fig. 1) to an uncapacitated one. We replace each capacitated arc (i, j) with an additional node k and two arcs (i, k) and (k, j) as shown in Fig. 2. The virtual node k is a demand node with $b(k) = u_{ij}$, the capacity of the original arc (i, j), and the demand of node j is reduced by u_{ij}. The transformed graph is bipartite and not connected, because there are no outgoing arcs from these virtual nodes.

To implement the RHS-scaling algorithm, we assume that all parameters in the original graph (demand at each node, cost and capacity associated with each arc) are integers. We let $U = \max\{\{b(i)\}_{i\in\mathcal{N}},\ \{u_{ij}\}_{(i,j)\in\mathcal{L}}\}$, which is the maximum demand in the transformed (uncapacitated) graph. We note that such integer approximation leads to negligible approximation error, if U is made sufficiently large (e.g., 2^{20}). It is also worth noting that the complexity of the RHS-scaling algorithm is logarithmic in U [47].

We associate each node i of the transformed (uncapacitated) graph a **potential** $\pi(i)$, and define the **reduced cost** of each arc (i, j) as

$$\bar{c}_{ij} := c_{ij} - \pi(i) + \pi(j). \tag{22}$$

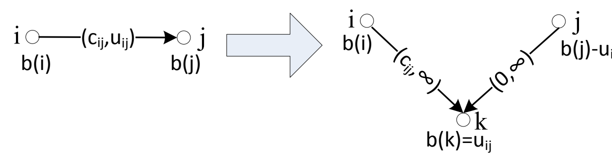

Fig. 2. Converting a capacitated arc into uncapacitated ones.

The basic idea of this algorithm is to augment Δ unit of flow from a supply node to a demand node, through the "shortest" path between them with the minimum total reduced cost. The parameter Δ is initially set to be $2^{\lceil \log U \rceil} \geq U$, and is replaced by $\Delta/2$ at each iteration. The vector $\mathbf{y}$ is a set of flows over the transformed graph, and the vector $\mathbf{e}$ records the residual demand at each node.

Algorithm RHS-SCALING
begin
  set $\mathbf{y} := \mathbf{0}$, $\mathbf{e} := \mathbf{b}$, $\boldsymbol{\pi} := \mathbf{0}$, and $\Delta := 2^{\lceil \log U \rceil}$;
  while there is an imbalanced node **do**
    $S(\Delta) = \{i : b(i) \leq -\Delta\}$;
    $T(\Delta) = \{i : b(i) > -\Delta\}$.
    while $S(\Delta) \neq \emptyset$ and $T(\Delta) \neq \emptyset$ **do**
      1. let $i \in S(\Delta)$ and $j \in T(\Delta)$;
      2. with the updated $\boldsymbol{\pi}$, compute the reduced cost $\bar{c}_{ij}$ for each arc according to (22);
      3. considering the reduced cost $\bar{c}_{ij}$ as the length of every arc (i, j), compute the shortest path distance from node i to every node k, $d(k)$;
      4. augment Δ units of flow along the shortest path from node i to node j;
      5. $\pi(k) := \pi(k) - d(k), \qquad \forall\, k$;
      6. update $\mathbf{y}$, $\mathbf{e}$, $S(\Delta)$, and $T(\Delta)$;
    end;
    $\Delta = \Delta/2$;

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end;
end;

Theorem *4.2:* Consider the setting with deterministic inelastic demand and electricity prices. The RHS-Scaling algorithm terminates with a feasible set of network flows that minimizes the total cost. Let $\mathbf{y}$ be the flow vector after the algorithm terminates. It determines an optimal storage operation policy[10], as well as the value of storage expressed in the following,

$$\sum_{t=0}^{T-1} p_t \ell_t - \sum_{(i,j)} y_{ij} c_{ij}, \tag{23}$$

where the second term is the sum over all arcs in the transformed (uncapacitated) graph. □

The optimality of the RHS-Scaling algorithm is proved in [47]; in Appendix E we also present a proof for completeness. Note that the first term in (23), $\sum_{t=0}^{T-1} p_t \ell_t$, is the consumer's total energy cost without storage, and the second term in (23) is the consumer's total energy cost if she optimally operates the storage (cf. Eq. (21)); the difference between these two costs, is therefore the **value of storage** (see its formal definition in the next section). The complexity of this algorithm increases linearly with the number of stages T, and scales in the order of $\log U$ [47].

V. Value of Storage

For the general setting formulated in Section II, we formally define the value of storage (VoS), and show that the VoS is a concave function of the storage capacity B. We then consider an important special case where the purchasing price always equals the selling price. We show that the VoS does not depend on the consumer's demand; in other words, it is optimal to operate the storage only for arbitrage. If we further relax the charging/dischharging rate constraints,[11] then the value of storage is shown to be linear in the storage capacity B.

The optimal payoff-to-go $V_0(0, s_0, f_0)$ is the maximum expected payoff the consumer could obtain (with a storage of capacity B), under an initial consumer state f_0 and an initial global state s_0. We note that $V_0(0, s_0, f_0)$ is closely related to the **value of storage** (VoS), which is defined as the difference between the maximum expected payoffs achieved by two consumers, the former of whom owns a storage of capacity B, and the latter does not own a storage. Formally, for a consumer with initial state f_0, the value of a capacity-B storage is defined by

$$\mathrm{VoS}(B, f_0) = \mathbb{E}\left\{V_0(0, s_0, f_0) - \underline{V}_0(0, s_0, f_0)\right\}, \tag{24}$$

where the expectation is over the initial global state s_0, and $\underline{V}_0(0, s_0, f_0)$ is the optimal payoff-to-go (of the dynamic program defined in Section II) with $B = 0$.

[10]Earlier in this section we have discussed how a set of feasible network flows corresponds to a feasible storage operation policy.

[11]The setting with no charging/dischharging rate constraints is motivated by the fact that fast-response storage devices are rapidly becoming available; for example, the lithium-ion titanate batteries are capable of recharging in approximately 10 minutes to 95% of full capacity [50].

Lemma 3.1 shows that the optimal payoff-to-go at stage t is concave in the vector (x_t, f_t) (a joint state of storage level and consumption status). Hence, the VoS defined in (24) can be calculated by solving a sequence of (deterministic) convex optimization problems (on the right hand side of (9)), after the discretization of the system space.

The following proposition shows that the VoS is concave in the storage capacity. In other words, the marginal value of storage decreases as the size of storage increases. Its proof is deferred to Appendix F.

Proposition 5.1: For every initial state f_0, the value of storage (defined in (24)) is a concave function of the storage capacity B. □

Next, we derive some strong results for **an important special case where the purchasing price always equals the selling price**, which is the setting used in [32] to study the optimal storage operation under random inelastic demand. The following proposition shows that the optimal storage operation (as well as the value of storage) is independent of the consumer's demand. It is worth noting that this result holds for a more general setting where the consumer's state transition is arbitrary, i.e., the next state f_{t+1} could be an arbitrary function of the current state f_t and the total consumption $c_t + d_t$ (not necessarily according to the linear transition defined in (4)).

Proposition 5.2: Suppose that the selling price always equals the purchasing price, i.e., $p_t(s_t) = q_t(s_t)$ for every t and $s_t \in \mathcal{S}_t$. The optimal policy is independent of the consumer's initial state f_0 and utility functions $\{u_t\}_{t=0}^{T-1}$. □

We prove Proposition 5.2 in Appendix G, by showing the existence of an optimal policy that uses the storage only for arbitrage. This practically useful result enables the consumer to compute the optimal storage operation policy according to Corollary 3.1 (by setting the utility function to be constantly zero), without taking into account her true energy demand.

VI. Numerical Examples

In this section, we present several numerical examples that compute the value of a finite-capacity storage under different parameter settings (e.g. the storage capacity and prices during peak hours). In Sections VI-A and VI-B, we will stick to the setting in Section IV (where the consumer is faced with deterministic prices and inelastic demand), and apply the RHS scaling algorithm to compute the optimal storage operation. In Section VI-C, we consider a more complicated setting with (Markov modulated) random prices.

In this section, we let each stage last for one hour. For simplicity, we will consider fast-charging storage devices that can be fully charged within one hour, i.e., $R^C = R^D = B$. We set the charging efficiency $\gamma = 0.85$, and the discharging efficiency $\eta = 0.85$ [51].

A. Storage Operation under Critical Peak Pricing

In this subsection, we consider a simple setting where the consumer faces two-level deterministic prices and inelastic demand. This simplified setting incorporates a variety of demand response programs that have been offered to end

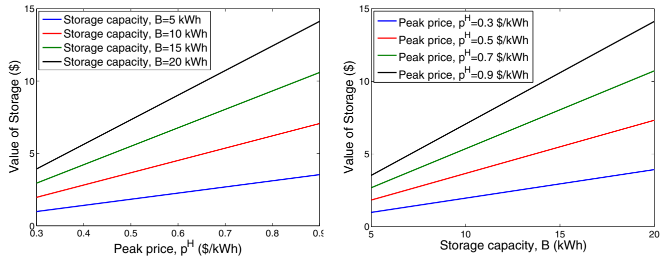

Fig. 3. Plot of the value of storage under different values of storage capacity and peak purchasing price.

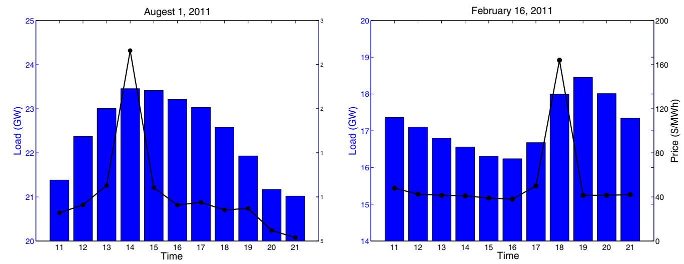

Fig. 4. Real-time prices and actual system load, ISO New England Inc. Blue bars represent the real-time system loads and the dots connected by a black line represent the hourly prices.

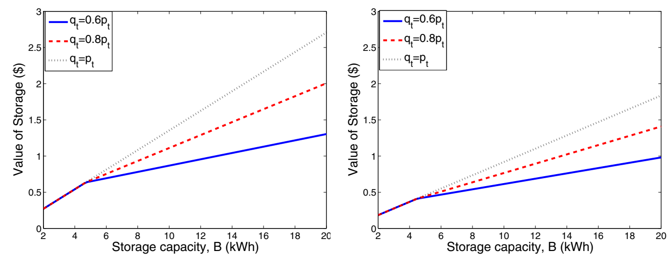

Fig. 5. The left (right) subplot is the value of storage for the 11-hour price trajectory depicted on the left (right, respectively) subplot of Fig. 4.

consumers, for example, the critical peak pricing[12] (CPP) used in California utilities to reduce peaks in consumer end-use loads.

The consumer is faced with a deterministic time-invariant price at all stages except one stage of peak period. Formally, let stage $\tau \in \{2, \dots, T-1\}$ denote the peak hour, and

$$p_\tau = p^H, \quad q_\tau = q^H; \quad p_t = p^L, \quad q_t = q^L, \quad \forall\, t \neq \tau.$$

Let ℓ_τ denote the inelastic demand during the peak hour.

Under critical peak pricing, the "critical peak price" during the peak hour is usually much higher than the normal price. We will therefore focus on the case with $p^L/(\gamma\eta) < q^H \leq p^H$. In this case, it is optimal to fully charge the battery at stage $\tau - 1$ and then fully discharge it during at stage τ.[13] The value of storage is given by

$$\min\{\ell_\tau, \eta B\} p^H - p^L B/\gamma + q^H \left(\eta B - \ell_\tau\right)^+, \tag{25}$$

where B/γ is the amount of energy procured at the off-peak period $\tau - 1$, $p^L B/\gamma$ is the payment made by the consumer, $\min\{\ell_\tau, \eta B\} p^H$ is the consumer's saving at the peak period τ, and finally, $q^H \left(\eta B - \ell_\tau\right)^+$ is the revenue earned by the consumer if she sells energy (back to the grid) at stage τ.

We assume that $p^L = q^L$ and $p^H = q^H$, i.e., the purchasing price always equals the selling price. In this case, we observe from Eq. (25) that the VoS is independent of the inelastic demand during the peak hour, ℓ_τ. This is in accordance with Proposition 5.2.

Let the off-peak price be $p^L = q^L = 0.05\$/\text{kWh}$. We consider the case where the peak-hour price is at least $0.3\$/\text{kWh}$. Since $p^L/(\gamma\eta) < q^H = p^H$, the value of storage is expressed in Eq. (25). We observe from both Eq. (25) and Fig. 3 that the value of storage increases linearly with the peak price p^H, as well as the storage capacity B.

B. Storage Operation under Day-ahead (hourly) Pricing

We now consider a more complicated setting where the consumer pays (deterministic) day-ahead hourly prices. Suppose that the consumer is faced with one of the two trajectories of day-ahead hourly prices presented in Fig. 4. If we refer to the hour starting at 11AM as stage 0, then the last hour (starting at 9PM) is stage 10, and $T = 11$ in this example. The ratio of the selling price to the purchasing price (at every stage t) is set to be 0.6, 0.8, and 1. Suppose that the consumer's hourly energy demand (on each of these two days) is inelastic and has the same shape as the hourly system load depicted in Fig. 4. We rescale the consumer's hourly demand such that her highest hourly demand (i.e., at stage 3 of Aug. 1 and stage 8 of Feb. 16) is normalized to be 4kWh.

Fig. 5 depicts the VoS under the two price trajectories and different (selling/purchasing) peak price ratios. For the case where the purchasing price equals the selling price, the VoS does not depend on the consumer's demand (cf. Proposition 5.2), and increases linearly with the capacity B, due to the lack of charging/discharging constraints. If the selling price is lower than the purchasing price, then the VoS becomes a piecewise-linear concave function of the capacity B. This is because when the storage capacity is no more than the (normalized)[14] demand at the peak hour (i.e., stage 3 of Aug. 1 and stage 8 of Feb. 16), the marginal VoS is the difference between the (normalized) purchasing price at the peak hour and that at earlier off-peak hours; when the storage capacity is higher than the (normalized) demand at the peak hour, on the other hand, the marginal VoS is determined by the *selling price* at the peak hour. We also note that the price trajectory on Aug. 1, 2011 yields a higher VoS because of the extremely high price during 2PM-3PM.

C. Storage Operation under Random Prices

In this subsection we explore the relationship between VoS and price volatility by considering the first trajectories of day-

[12]Although contracts offered by different utilities may vary, consumers participating in a CPP program typically face higher electricity prices during critical peak periods (e.g., one hour in a hot summer afternoon), and will receive notice (from utility companies) one day before the peak hour.

[13]If $p^L/(\gamma\eta) \geq p^H$ then it is optimal not to charge the storage. Otherwise, if $q^H \leq p^L/(\gamma\eta) \leq p^H$ then it is optimal to charge the storage to level $\min\{\ell_\tau/\eta, B\}$ at stage $\tau - 1$ and withdraw the storage only for consumption at stage τ.

[14]The "normalized" demand and purchasing price take into account the discharging efficiency and/or the charging efficiency.

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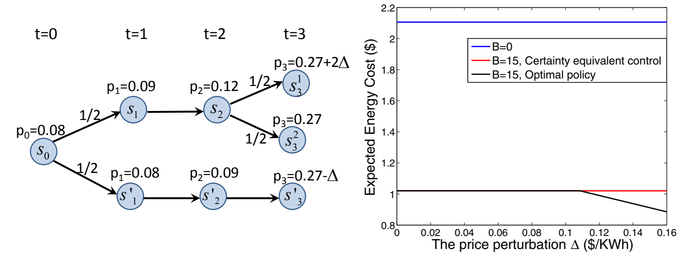

Fig. 6. The left subplot shows the information structure on the random purchasing prices (in \$/kWh). The right subplot compares the total energy costs resulting from optimal storage operation and certainty-equivalent control.

ahead hourly prices (presented in the left subplot of Fig. 4) with random perturbation. As in Section VI-B, the consumer's hourly demand is assumed to be inelastic and have the same shape as the hourly system load depicted on the left subplot of Fig. 4. Again, the consumer's hourly demand is rescaled such that her highest hourly demand (at stage 3) is 4kWh. In this subsection we fix the storage capacity B to be 15kWh, and let the ratio of the selling price to the purchasing price (at every stage t) be 0.6.

The purchasing prices at stages $4-10$ are deterministic and are shown in the left subplot of Fig. 4. For stages $0-3$, the random purchasing prices are determined by the global state. The evolution of the global state s_t (as well as the purchasing price $p_t(s_t)$) is depicted in the left subplot of Fig. 6. At the beginning of every stage t, the consumer observes the realized global state, and takes an action to maximize her expected payoff. There is only one global state at stage 0, and new price estimate becomes available to the consumer at the beginning of stage 1. With probability $1/2$ the global state at stage 1 is s_1, which leads to a higher purchasing price of 0.09\$/kWh at stage 1; following the global state s_1, with probability $1/2$ we move to the global state s_3^1 at stage 3, and with probability $1/2$ the global state at stage 3 is realized as s_3^2. With probability $1/2$ the global state at stage 1 is s_1', which will lead to a lower purchasing price at stage 3 with probability one. In the left subplot of Fig. 6, $\Delta \in [0, 0.16]$ is a parameter reflecting the volatility of the purchasing price in the peak hour.

For a given Δ, we compute the optimal two-threshold policy characterized in Corollary 3.1 through backward induction. When $\Delta \leq 0.1085$, it is optimal to fully charge the storage at stage 0, regardless of the global state realized at stage 1; in the language of Corollary 3.1, we have $h_0(s_0) = B$, i.e., it is optimal to greedily charge the storage up to level B at stage 0. At stage 3, the consumer withdraws all the stored energy for consumption and selling back to the grid.

When $\Delta > 0.1085$, the optimal threshold policy charges the storage up to level ℓ_3/η at stage 0 (here, ℓ_3 is the consumer's inelastic demand at stage 3). Then, at stage 1, the consumer fully charges the storage only at the global state s_1. This is because at the global state s_1' it is not worth charging the storage so as to sell the stored energy at stage 3. When $\Delta > 0.1085$, it is optimal to fully charge the storage only at state s_1, due to the low selling price at state s_3'. In the language of Corollary 3.1, when $\Delta > 0.1085$ we have $h_0(s_0) = \ell_3/\eta$, $h_1(s_1) = B$, and $h_1(s_1') = \ell_3/\eta$.

In the right subplot of Fig. 6 we compare the expected energy cost (i.e., the consumer's total expected payment minus her total expected revenue in stages $0-3$) resulting from the optimal threshold policy and a certainty equivalent heuristic policy. The latter policy solves a one-shot optimization problem to maximize the consumer's payoff, where all random variables (e.g., electricity prices) take their expected values. In our setting, this certainty equivalent policy fully charges the storage at stage 0 and withdraws all the stored energy (for consumption and selling back to the grid) at stage 3. As demonstrated in the right subplot of Fig. 6, the certainty equivalent policy is optimal when $\Delta \leq 0.1085$; for larger Δ the certainty equivalent policy results in up to 14% more total cost than the optimal policy. We note that in the right subplot of Fig. 6, the gap between the blue and the black curves is the *value of storage*, i.e., the consumer's expected energy saving resulting from the optimal operation of the storage.

Remark *6.1 (Investment v.s. Value of storage):* Before ending this section, we would like to make some brief discussion on the practical implication of our numerical results. We note that the estimated VoS (under the pricing schemes used in Sections VI-B and VI-C) is comparable to the one-time setup cost (on energy storage equipment) of a residential consumer. For example, consumers can obtain a 10kWh Tesla battery system (from SolarCity, an American provider of energy services) under a 10-year lease for a \$1500 down payment plus \$15 per month (\$3300 in total over 10 years).[15] That amounts to less than \$1 fixed cost per day. We emphasize here that the actual value of storage and the optimal sizing of storage depend heavily on the spread and volatility of electricity prices as well as the energy consumption profile of the consumer, and could be very different from the estimates made in this section. ■

VII. Conclusion and Future Work

We study the optimal operation and economic value of energy storage at consumer locations, through a dynamic programming formulation. For a general setting that incorporates consumer inter-temporal energy demand, we prove the optimality of a two-threshold policy.

We show that the computation of the characterized optimal threshold policy could be significantly simplified when the consumer's demand is inelastic. For a setting with random (inelastic) demand and stochastic electricity prices, we show that the consumer's maximum expected payoff is piecewise linear in the storage level. If both the energy demand and electricity prices are deterministic, we establish the equivalence between the optimal storage operation problem and a minimum cost flow problem that can be easily solved by greedy algorithms of linear complexity.

We define the value of storage (VoS) as the consumer's expected net benefit if she optimally operates the storage. We show that the value of storage is a concave function of storage capacity. If the consumer can always buy and sell electricity at the same (realized) price, then it is optimal to use the storage only for arbitrage. As a result, the optimal operation

[15] http://www.solarcity.com/residential/energy-storage

of the energy storage as well as the VoS is independent of the consumer's demand.

There are a variety of interesting directions for future work. For example, it would be interesting to study the cooperative operation of multiple storage devices, for an aggregator who bids into the wholesale electricity market. Another interesting direction is to extend the DP framework constructed in this work to incorporate more types of electricity loads (e.g., uninterruptible loads) and nonlinear pricing schemes.

Appendix A
Proof of Lemma 3.1

We will prove the lemma through backward induction. For the terminal stage T, the desired result holds. Suppose that at stage $t+1$, the optimal payoff-to-go is concave in (x_{t+1}, f_{t+1}), for every s_{t+1}. We will show that the optimal payoff-to-go function at stage t is concave in (x_t, f_t), regardless of s_t. We first have

$$V_t(x_t, s_t, f_t) = \max_{(a_t,c_t,d_t)} \{ w_t(x_t, s_t, f_t, a_t, c_t, d_t) + \mathbb{E}\left[V_{t+1}(x_{t+1}, s_{t+1}, f_{t+1})\right] \}. \tag{26}$$

To prove that $V_t(x_t, s_t, f_t)$ is concave in (x_t, f_t), we will show that

$$\frac{V_t(x_t, s_t, f_t) + V_t(x_t', s_t, f_t')}{2} \le V_t\left(\tilde{x}_t, s_t, \tilde{f}_t\right), \quad \forall\ (x_t, f_t),\ (x_t', f_t') \in [0, B] \times [0, Z], \tag{27}$$

where $\tilde{x}_t = (x_t + x_t')/2$ and $\tilde{f}_t = (f_t + f_t')/2$. Let μ_t^* be an optimal decision making rule at stage t, $\mu_t^*(x_t, s_t, f_t) = (a, c, d)$, and $\mu_t^*(x_t', s_t, f_t') = (a', c', d')$. It follows from (5) that the following action

$$(\tilde{a}, \tilde{c}, \tilde{d}) \triangleq ((a + a')/2, (c + c')/2, (d + d')/2),$$

is feasible under the state $(\tilde{x}_t, s_t, \tilde{f}_t)$. It follows from the concavity of the utility function $u_t(\cdot)$ that

$$\begin{aligned} &w_t(x_t, s_t, f_t, a, c, d) + w_t(x_t', s_t, f_t', a', c', d') \\ &\le 2w_t(\tilde{x}_t, s_t, \tilde{f}_t, \tilde{a}, \tilde{c}, \tilde{d}). \end{aligned} \tag{28}$$

Let (x_{t+1}, f_{t+1}), (x_{t+1}', f_{t+1}'), and $(\tilde{x}_{t+1}, \tilde{f}_{t+1})$ denote the storage level and consumer state at stage $t+1$, under the original states (x_t, f_t), (x_t', f_t'), $(\tilde{x}_t, \tilde{f}_t)$ and actions (a_t, c_t, d_t), (a_t', c_t', d_t'), $(\tilde{a}_t, \tilde{c}_t, \tilde{d}_t)$ taken at stage t, respectively. We have

$$(\tilde{x}_{t+1}, \tilde{f}_{t+1}) = \frac{(x_{t+1}, f_{t+1}) + (x_{t+1}', f_{t+1}')}{2}.$$

The concavity of the value function at stage $t+1$ implies that

$$\begin{aligned} &\mathbb{E}\{V_{t+1}(x_{t+1}, s_{t+1}, f_{t+1})\} + \mathbb{E}\{V_{t+1}(x_{t+1}', s_{t+1}, f_{t+1}')\} \\ &\le 2\mathbb{E}\{V_{t+1}(\tilde{x}_{t+1}, s_{t+1}, \tilde{f}_{t+1})\}, \end{aligned} \tag{29}$$

where the expectation is over the next global state s_{t+1}. Inequalities in (28) and (29) imply that

$$\begin{aligned} &\left(V_t(x_t, s_t, f_t) + V_t(x_t', s_t, f_t)\right)/2 \\ &\le w_t(\tilde{x}_t, s_t, \tilde{f}_t, \tilde{a}, \tilde{c}, \tilde{d}) + \mathbb{E}\{V_{t+1}(\tilde{x}_{t+1}, s_{t+1}, \tilde{f}_{t+1})\} \\ &\le V_t(\tilde{x}_t, s_t, \tilde{f}_t), \end{aligned}$$

where the last inequality follows from (26).

Appendix B
Proof of Theorem 3.1

In order to prove part (b), we will show that given any $x_t \in [h_t(s_t, f_t), k_t(s_t, f_t)]$, the action $(a_t = 0, c_t = y_t(\mathbf{z}_t), d_t = 0)$ satisfies the first order conditions in (12), and is therefore optimal.

Under the assumption on the incremental salvage value that $w_T(x_T', s_T, f_T) - w_T(x_T, s_T, f_T)$ is non-decreasing in f_T for any $x_T' > x_T$, and that $w_T(x_T, s_T, f_T') - w_T(x_T, s_T, f_T)$ is non-decreasing in x_T for any $f_T' > f_T$, it is straightforward to check that for every t and every s_t, the partial derivative of $\bar{V}_{t+1|s_t}(x, f)$ with respect to x is non-decreasing in f, and that the partial derivative of $\bar{V}_{t+1|s_t}(x, f)$ with respect to f is non-decreasing in x. It follows from the first-order conditions in (12) that given any (s_t, f_t), $y_t(\mathbf{z}_t)$ is non-increasing in x_t, due to the facts that $\beta_t \le 0$ and that u_t' is concave in $c_t + d_t$. Here, and in the rest of the proof, we let $\mathbf{z}_t = (x_t, s_t, f_t)$.

We fix (s_t, f_t) and consider a storage level $x_t \in [h_t(s_t, f_t), k_t(s_t, f_t)]$. For notational convenience, we let $x_t' = k_t(s_t, f_t) \ge x_t$. We will show that the action $(a_t = 0, c_t = y_t(\mathbf{z}_t), d_t = 0)$ satisfies the third and fourth condition in (12) (with respect to d_t). We first argue that

$$\begin{aligned} &\partial_f^- \bar{V}_{t+1|s_t}(x_t', \alpha_t f_t + \beta_t y_t(\mathbf{z}_t') + \vartheta_t(s_t)) \\ &\ge \partial_f^+ \bar{V}_{t+1|s_t}(x_t, \alpha_t f_t + \beta_t y_t(\mathbf{z}_t) + \vartheta_t(s_t)), \end{aligned} \tag{30}$$

where $\mathbf{z}_t' = (x_t', s_t, f_t)$. We note that $y_t(\mathbf{z}_t') \le y_t(\mathbf{z}_t)$. If $y_t(\mathbf{z}_t) > 0$, then from the second inequality in (12) we have

$$\begin{aligned} &u_t'(y_t(\mathbf{z}_t), s_t, f_t) - p_t(s_t) \\ &\ge -\beta_t \partial_f^+ \bar{V}_{t+1|s_t}(x_t, \alpha_t f_t + \beta_t y_t(\mathbf{z}_t) + \vartheta_t(s_t)). \end{aligned} \tag{31}$$

The first inequality in (12) implies that

$$\begin{aligned} &u_t'(y_t(\mathbf{z}_t'), s_t, f_t) - p_t(s_t) \\ &\le -\beta_t \partial_f^- \bar{V}_{t+1|s_t}(x_t', \alpha_t f_t + \beta_t y_t(\mathbf{z}_t') + \vartheta_t(s_t)). \end{aligned} \tag{32}$$

Since $y_t(\mathbf{z}_t') \le y_t(\mathbf{z}_t)$ and the utility function is concave in its first argument, the above two inequalities in (31) and (32) imply the desired result in (30). For the case with $y_t(\mathbf{z}_t) = 0$, we must have $y_t(\mathbf{z}_t') = 0$, and the inequality in (30) follows from the fact that the partial derivative of $\bar{V}_{t+1|s_t}(x, f)$ with respect to f is non-decreasing in x.

Since $y_t(\mathbf{z}_t') \le y_t(\mathbf{z}_t)$ and $\beta \in [-1, 0]$, we have

$$\begin{aligned} f_{t+1} &:= \alpha_t f_t + \beta_t y_t(\mathbf{z}_t) + \vartheta_t(s_t) \\ &\le \alpha_t f_t + \beta_t y_t(\mathbf{z}_t') + \vartheta_t(s_t) := f_{t+1}'. \end{aligned} \tag{33}$$

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Hence, the concavity of $V_{t+1|s_t}(x_{t+1}, f_{t+1})$ implies that

$$\begin{aligned}(f'_{t+1} - f_{t+1})\partial_f^- \bar{V}_{t+1|s_t}(x'_t, f'_{t+1})& \\ +(x'_t - x_t)\partial_x^- \bar{V}_{t+1|s_t}(x'_t, f'_{t+1})& \\ \leq (f'_{t+1} - f_{t+1})\partial_f^+ \bar{V}_{t+1|s_t}(x_t, f_{t+1})& \\ +(x'_t - x_t)\partial_x^+ \bar{V}_{t+1|s_t}(x_t, f_{t+1}).&\end{aligned} \tag{34}$$

Combining (30) and (34) we have

$$\partial_x^- \bar{V}_{t+1|s_t}(x'_t, f'_{t+1}) \leq \partial_x^+ \bar{V}_{t+1|s_t}(x_t, f_{t+1}), \tag{35}$$

i.e., the marginal value of storage is higher under the lower storage level x_t and the corresponding action $(a_t = 0, c_t = y_t(\mathbf{z}_t), d_t = 0)$. Here, and in the rest of this proof, f_{t+1} and f'_{t+1} are notations defined in (33).

Since $x'_t = k_t(s_t, f_t) \geq h_t(s_t, f_t)$, it follows from the monotonicity result in (35) that x'_t (together with f'_{t+1}) satisfies the condition in (14). It follows from the definition of $k_t(s_t, f_t)$ that the condition in (13) holds at storage level x'_t. We conclude from Lemma 3.2 that the action $(a_t = 0, c_t = y_t(\mathbf{z}'_t), d_t = 0)$ is optimal under the system state (x'_t, s_t, f_t). As a result, it is optimal to have $d_t = 0$ under storage level $x_t \in [h_t(s_t, f_t), k_t(s_t, f_t)]$, because $d_t = 0$ satisfies the third and fourth conditions in (12) (due to the definition of $k_t(s_t, f_t)$ and the inequality in (35)).

We note that the action $(a_t = 0, c_t = y_t(\mathbf{z}_t), d_t = 0)$ must also satisfy the last two conditions in (12), i.e., it is optimal not to sell the stored energy back to grid, because the marginal value of storage at $(x_{t+1} = x_t, f_{t+1})$ is no less than that at $(x_{t+1} = x'_t, f'_{t+1})$ (according to Eq. (35)).

Since the condition in (14) holds for the storage level $h_t(s_t, f_t)$ and $x_t \geq h_t(s_t, f_t)$, it follows from (35) that the condition in (14) holds for x_t. Therefore, the action $(a_t = 0, c_t = y_t(\mathbf{z}_t), d_t = 0)$ satisfies the fifth and sixth conditions in (12), i.e., it is optimal not to charge the storage. So far we have proved part (b).

We now prove part (a). We note that the monotonicity result in (35) indeed holds for any $0 \leq x_t \leq x'_t$. Given a storage level $x_t < h_t(s_t, f_t) \leq k_t(s_t, f_t)$, the condition in (14) must be violated at the action $(a_t = 0, c_t = y_t(\mathbf{z}_t), d_t = 0)$, and therefore it is optimal to charge the storage (see the fifth and sixth conditions in Eq. (12)).

Analogously, if $x_t > k_t(s_t, f_t)$, then the condition in (13) must be violated at $(a_t = 0, c_t = y_t(\mathbf{z}_t), d_t = 0)$, and therefore it is optimal to withdraw the storage for consumption or selling back to the grid. Part (c) of the theorem holds.

Appendix C
Proof of Corollary 3.1

For part (a), it is straightforward to show that if the storage level x_t is below the threshold $h_t(s_t)$, then it is optimal to greedily charge the storage up to this threshold, at which the marginal value of storage equals the purchasing price $p_t(s_t)$. Part (b) follows from Part (a) and Lemma 3.2.

Now consider the case where the storage level x_t is above the threshold $k_t(s_t)$. If $u'_t(0, s_t) \leq q_t(s_t)$, then the consumer should sell (instead of consuming) the stored electricity, and therefore we have $d^*_t = 0$ in (18). If $u'_t(0, s_t) > q_t(s_t)$, the consumer withdraws the storage for consumption, until at least one of the following conditions holds:

1) the discharging rate constraint is met, or the storage becomes empty, i.e., $d_t \leq \eta \min\{R^D, x_t\}$;
2) the marginal utility equals the adjusted selling price, i.e., $d_t \leq e_t(s_t)$;
3) the marginal storage value (at the next stage $t + 1$) is no less than either the marginal utility or the purchasing price, i.e.,
$$\partial_x^+ \bar{V}_{t+1}(x_t - d_t/\eta) \leq \eta \min\{u'_t(d_t, s_t), p_t(s_t)\},$$
where $x_t - d_t/\eta$ is un upper bound on x_{t+1}, since $d_t > 0$ implies $a_t \leq 0$.

After consuming d^*_t amount of energy from the storage, if the marginal storage value (at stage $t+1$) remains lower than the adjusted selling price, i.e., if

$$x_t - d^*_t/\eta > g_t(s_t),$$

then sell $(a^*_t)^-$ amount of stored electricity to the grid, until the discharging rate limit is reached $\left(d^*_t + (a^*_t)^-\right)/\eta = R^D$, or the marginal value of storage at stage $t + 1$ equals the adjusted selling price, i.e.,

$$x_{t+1} = x_t - \left(d^*_t + (a^*_t)^-\right)/\eta = g_t(s_t).$$

After consuming d^*_t amount of energy from the storage, if the marginal utility $u'_t(d^*_t, s_t)$ is higher than the purchasing price $p_t(s_t)$, then the consumer should purchase electricity for consumption up to the level $y_t(s_t)$ (cf. its definition in (16)), i.e., $c^*_t = (y_t(s_t) - d^*_t)^+$.

Appendix D
Proof of Theorem 4.1

We first note that the concavity of $\bar{V}_{t+1|s_t}(x_{t+1})$ follows from the concavity of $V_{t+1}(x, s)$ in x (cf. Lemma 3.1), and the definition of $\bar{V}_{t+1|s_t}(x_{t+1})$ in Eq. (10).

In what follows, we prove that $\partial^\pm \bar{V}_{t+1|s_t}(x_{t+1})$ are step functions, i.e., for $t = 0, \ldots, T-1$,

$$\partial^+ \bar{V}_{t+1|s_t}(x) = \begin{cases} v_1, & x \in [b_0, b_1), \\ \ldots, & \\ v_i, & x \in [b_{i-1}, b_i), \\ \ldots, & \\ v_n, & x \in [b_{n-1}, b_n), \end{cases} \tag{36}$$

where b_i is increasing in i with $b_0 = 0$, $b_n = B$, and v_i is decreasing in i, and

$$\partial^- \bar{V}_{t+1|s_t}(x) = \begin{cases} v_1, & x \in (b_0, b_1], \\ \ldots, & \\ v_i, & x \in (b_{i-1}, b_i], \\ \ldots, & \\ v_n, & x \in (b_{n-1}, b_n]. \end{cases} \tag{37}$$

We will prove this result by induction. It is straightforward to check that $\bar{V}_{T|s_{T-1}}(x_T)$ is piecewise linear, since the salvage value is linear (cf. (19)) and $\bar{V}_{T|s_{T-1}}(x_T)$ is a convex combination of these linear functions (cf. Eq. (10)). It follows that $\partial^\pm \bar{V}_{T|s_{T-1}}(x_T)$ are step functions.

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Suppose that $\partial^{\pm}\bar{V}_{t+1|s_t}(x_{t+1})$ are step functions of the forms in (36), (37), we now argue that $\partial^{\pm}\bar{V}_{t|s_t}(x_t)$ are step functions. For notational convenience, we define

$$\bar{a}_t = \min\left\{R^C, h_t(s_t)\right\}/\gamma,$$

and

$$\bar{d}_t = \eta \min\left\{B - k_t(s_t), \ell_t(s_t)/\eta, R^D\right\}.$$

The characterization of an optimal policy in Corollary 3.1 enables us to establish the following relations between $\partial^{\pm}V_t(x, s_t)$ and $\partial^{\pm}\bar{V}_{t+1|s_t}(x)$:

$$\begin{aligned} &\partial^+ V_t(x, s_t) \\ &= \begin{cases} \partial^+\bar{V}_{t+1|s_t}(x + \gamma\bar{a}_t), & x \in [0, h_t(s_t) - \gamma\bar{a}_t), \\ p_t(s_t)/\gamma, & x \in [h_t(s_t) - \gamma\bar{a}_t, h_t(s_t)), \\ \partial^+\bar{V}_{t+1|s_t}(x), & x \in [h_t(s_t), k_t(s_t)), \\ \eta p_t(s_t), & x \in \left[k_t(s_t), k_t(s_t) + \bar{d}_t/\eta\right), \\ \partial^+\bar{V}_{t+1|s_t}(x - \bar{d}_t/\eta), & x \in \left[k_t(s_t) + \bar{d}_t/\eta, g_t(s_t)\right), \\ \eta q_t(s_t), & \\ \quad x \in \left[g_t(s_t), \min\{B, g_t(s_t) + (R^D - \bar{d}_t/\eta)\}\right), & \\ \partial^+\bar{V}_{t+1|s_t}(x - R^D), & \\ \quad x \in \left[\min\{B, g_t(s_t) + (R^D - \bar{d}_t/\eta)\}, B\right); & \end{cases} \end{aligned} \tag{38}$$

and

$$\begin{aligned} &\partial^- V_t(x, s_t) \\ &= \begin{cases} \partial^-\bar{V}_{t+1|s_t}(x + \gamma\bar{a}_t), & x \in (0, h_t(s_t) - \gamma\bar{a}_t], \\ p_t(s_t)/\gamma, & x \in (h_t(s_t) - \gamma\bar{a}_t, h_t(s_t)], \\ \partial^-\bar{V}_{t+1|s_t}(x), & x \in (h_t(s_t), k_t(s_t)], \\ \eta p_t(s_t), & x \in \left(k_t(s_t), k_t(s_t) + \bar{d}_t/\eta\right], \\ \partial^-\bar{V}_{t+1|s_t}(x - \bar{d}_t/\eta), & x \in \left(k_t(s_t) + \bar{d}_t/\eta, g_t(s_t)\right], \\ \eta q_t(s_t), & \\ \quad x \in \left(g_t(s_t), \min\{B, g_t(s_t) + (R^D - \bar{d}_t/\eta)\}\right], & \\ \partial^-\bar{V}_{t+1|s_t}(x - R^D), & \\ \quad x \in (\min\{B, g_t(s_t) + (R^D - \bar{d}_t/\eta)\}, B]. & \end{cases} \end{aligned} \tag{39}$$

We have proved that $\partial^{\pm}V_t(x, s_t)$ are step functions. $V_t(x, s_t)$ therefore is piecewise linear in x, and as a result, $\partial^{\pm}\bar{V}_{t|s_{t-1}}(x)$, a convex combination of $V_t(x, s_t)$ over all $s_t \in \mathcal{S}_t$, must also be piecewise linear in x.

Appendix E
Proof of Theorem 4.2

We first convert the transformed (uncapacitated) graph into a "modified" graph with nonnegative arc costs. We replace the cost of each arc (i, j) by $c_{ij} + D(i) - D(j)$, where $D(i)$ is the shortest path distance from node i to all the other nodes on the transformed graph, if we take the arc cost c_{ij} as the length of arc (i, j). It is straightforward to see that $c_{ij} + D(i) - D(j) \geq 0$, and that the newly constructed "modified" graph yields the same optimal solution as the transformed graph. We further note that this modified graph and the uncapacitated graph (transformed from the original graph presented in Fig. 1) lead to exactly the same operation of the RHS-Scaling algorithm, because the demand at each node and the shortest path between every pair of connected nodes remain the same. It is therefore sufficient to show the optimality of the RHS-Scaling algorithm on the modified graph with nonnegative arc costs.

Since the cost of every arc is nonnegative, a feasible set of flows $\mathbf{y}$ and a set of potentials $\boldsymbol{\pi}$ constitute an optimal solution, if the following (linear programming) optimality condition holds for every arc (i, j) [47]:

$$\begin{cases} \bar{c}_{ij} = c_{ij} - \pi(i) + \pi(j) = 0, & \text{if } y_{ij} > 0, \\ \bar{c}_{ij} = c_{ij} - \pi(i) + \pi(j) \geq 0, & \text{if } y_{ij} = 0. \end{cases} \tag{40}$$

If there exist a set of potentials $\boldsymbol{\pi}$ that make Condition (40) hold for a (possibly non-feasible) flow vector $\mathbf{y}$, we say $\mathbf{y}$ is **dual feasible**. Therefore a flow vector $\mathbf{y}$ is optimal if it is feasible and dual feasible.

We initially set $\boldsymbol{\pi} = \mathbf{0}$, which makes the initial flow vector $\mathbf{y} = \mathbf{0}$ dual feasible. Starting with a dual feasible flow vector $\mathbf{y}$, we argue that the augmentation (defined in Step 3) preserves dual feasibility, if node potentials are updated according to Step 4 in the algorithm, i.e.,

$$\pi(k) = \pi(k) - d(k), \qquad \forall\, k, \tag{41}$$

where $d(k)$ is the shortest path distance from the supply node i (picked up in Step 1) to node k (here we take the reduced arc costs defined in (22) as arc lengths). Since the original flow vector $\mathbf{y}$ is dual feasible, we have the following.

1) For every arc (k, h) with $y_{kh} > 0$ we have $\bar{c}_{kh} = 0$, which implies that $d(k) = d(h)$. As a result, $\bar{c}_{kh}$ remains zero after the update (of $\boldsymbol{\pi}$) defined in Eq. (41).
2) For every arc (k, h) with $y_{kh} = 0$, we have $\bar{c}_{kh} \geq 0$. If the flow on this arc remains zero after the augmentation, then after the update of $\boldsymbol{\pi}$ we have
$$\bar{c}_{kh} := \bar{c}_{kh} + d(k) - d(h) \geq 0,$$
where $\bar{c}_{kh}$ on the right hand side is calculated according to the original node potentials $\pi(k)$ and $\pi(h)$, and the inequality follows from the definition of shortest paths.
3) For every arc (k, h) with $y_{kh} = 0$, if the flow on this arc becomes positive after the augmentation, then the arc (k, h) must be along the shortest path from the supply node i to node j (cf. Step 3 in the algorithm). As a result, after the update of $\boldsymbol{\pi}$ we have
$$\bar{c}_{kh} := \bar{c}_{kh} + d(k) - d(h) = 0,$$
where $\bar{c}_{kh}$ on the right hand side is calculated according to the original node potentials $\pi(k)$ and $\pi(h)$, and the equality holds because the difference between the shortest path distances $d(h)$ and $d(k)$ is exactly the original $\bar{c}_{kh}$.

We have shown that the pair $(\mathbf{y}, \boldsymbol{\pi})$ after every iteration is dual feasible. The optimality of the algorithm then follows from the fact that it terminates when all nodes are balanced; i.e., the final flow vector $\mathbf{y}$ is feasible and dual feasible, and is therefore optimal. The value of storage is the difference between the minimum cost of a feasible flow vector and the consumer cost without storage, as given by Eq. (23).

Appendix F
Proof of Proposition 5.1

Under a sequence of realized global states $\mathbf{s} = (s_0, \ldots, s_T)$, let $V_0^B(0, \mathbf{s}, f_0)$ denote the consumer payoff realized by an optimal policy π^B (for the operation of a storage with capacity B) that maximizes the *expected* consumer payoff. The VoS defined in (24) can be written by

$$\mathrm{VoS}(B, f_0) = \sum_{\mathbf{s}} \mathbb{P}(\mathbf{s}) V_0^B(0, \mathbf{s}, f_0) - \mathbb{E}_{s_0}\left\{V_0^B(0, s_0, f_0)\right\},$$

where $\mathbb{P}(\mathbf{s})$ denotes the probability that $\mathbf{s}$ is realized. To prove the concavity of the VoS, it is sufficient to show that $V_0^B(0, \mathbf{s}, f_0)$ is concave in B, for every $\mathbf{s}$ and f_0. We will show that

$$\left(V_0^B(0, \mathbf{s}, f_0) + V_0^{B'}(0, \mathbf{s}, f_0)\right)/2 \leq V_0^{\tilde{B}}(0, \mathbf{s}, f_0), \quad \forall\, 0 < B < B', \quad \forall\, \mathbf{s}, \quad \forall\, f_0, \tag{42}$$

where $\tilde{B} = (B + B')/2$. Fixing a sequence of realized global states $\mathbf{s}$, we let (a_t, c_t, d_t) denote the action taken by the optimal policy π^B at stage t, and (a'_t, c'_t, d'_t) denote the action taken by the optimal policy $\pi^{B'}$ at stage t. Since all constraints as well as the state transition of f_t is linear, the action sequence

$$\left\{\left((a'_t + a_t)/2, (c'_t + c_t)/2, (d'_t + d_t)/2\right)\right\}_{t=0}^{T-1} \tag{43}$$

is feasible in the problem with storage capacity $\tilde{B} = (B + B')/2$, and results in a sequence of consumer states that equal the average of f_t and f'_t (consumer states resulting from the policy π^B and $\pi^{B'}$, respectively), for every t and every initial state f_0. Due to the concavity assumption that $V_T(x_T, s_T, f_T)$ is concave in the vector (x_T, f_T), and that the utility function $u_t(x_t, s_t, f_t)$ is concave in the vector (x_t, f_t), for $t = 0, \ldots, T-1$ and every s_t, we conclude that the action sequence defined in (43) achieves an ex-post payoff no less than

$$\left(V_0^B(0, \mathbf{s}, f_0) + V_0^{B'}(0, \mathbf{s}, f_0)\right)/2,$$

for every $\mathbf{s}$ and f_0. The desired result in (42) follows from the fact that the maximum consumer payoff $V_0^{\tilde{B}}(0, \mathbf{s}, f_0)$ cannot be less than the payoff achieved by the action sequence (43).

Appendix G
Proof of Proposition 5.2

To prove the desired result, we will show the existence of an optimal policy that never withdraws energy from storage for consumption. At stage t, consider an arbitrary policy π that withdraws the storage for consumption, i.e., $d_t > 0$. Note that there exists an optimal policy that never charges and discharges the storage simultaneously, i.e., under the optimal policy $d_t > 0$ implies $a_t \leq 0$. We will therefore assume, without loss of generality, that under the policy π, $d_t > 0$ and $a_t \leq 0$. Consider a modified policy such that

$$\tilde{a}_t = a_t - d_t, \quad \tilde{c}_t = c_t + d_t, \quad \tilde{d}_t = 0.$$

The modified policy withdraws the same amount of energy from the storage, i.e.,

$$a_t^- + d_t = -a_t + d_t = -\tilde{a}_t + \tilde{d}_t = \tilde{a}_t^- + \tilde{d}_t,$$

and results in the same energy consumption, i.e.,

$$c_t + d_t = \tilde{c}_t + \tilde{d}_t.$$

Since the selling price and purchasing price are always the same, this modified policy yields the same stage payoff (expressed in Eq. (6)) as the original policy π at t. It follows from the state transition rule (of x_t and f_t) that both the original and the modified policies result in the same state in the next stage, (x_{t+1}, f_{t+1}). Since the evolution of the global state s_t is assumed to be exogenous (independent of the action taken by the consumer), both policies lead to the same future system dynamics, as well as the same long-term expected payoff for the consumer. We therefore conclude the existence of an optimal policy that never withdraws the storage for consumption, under which we always have $d_t = 0$.

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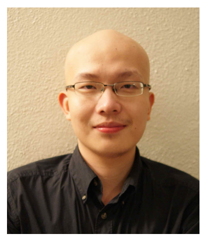

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